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What´s new and useful about chaos in Economic Science

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Abstract

Complexity is one of the most important characteristic properties of the economic behaviour. The new field of knowledge called Chaotic Dynamic Economics born precisely with the objective of understanding, structuring and explaining in an endogenous way such complexity. In this paper, and after scanning the principal concepts and techniques of the chaos theory, we analyze, principally, the different areas of Economic Science from the point of view of complexity and chaos, the main and most recent researches, and the present situation about the results and possibilities of achieving an useful application of those techniques and concepts in our field.

Keywords: Chaos, Complexity, Nonlinearities, Theory of Catastrophes, Lyapunov Exponents, Fractals, Attractors, Bifurcation, Time Series, Power Laws, Urban Dynamics, Chaotic Modelling, Artificial Neural Networks, Controlling Chaos, Predictability.

INTRODUCTION

Perhaps the most clear and ancient definition of chaos can be found in the words of Giordano Bruno written in his book published in 1583 in Venezia whit the title "De l´infinito universo mondi*": Now more than never I perceive that a tiny error in the beginning causes a big difference and a serious deviation at the end; a single problem was multiplied gradually branching out into an infinite number of other, just as a root spreads in infinite branches and masses.* Also, centuries later, in 1908, appears the G. K, Chesterton´s celebrated novel "The Man Who was Thursday", and in the page 12 said: *Why do all the clerks and navvies in the railway trains look so sad and tired, so very sad and tired?.I will tell you. It is because they know that the train is going right. It is because they know that whatever place they have taken a ticket for, that place they will reach. It is because after they have passed Sloane Square they know that the next station must be Victoria, and nothing but Victoria. Oh, their wild rapture! Oh, their eyes like stars and their souls again in Eden, if the next station were unaccountably Baker Street.* These two relevant references are indeed very clear and rich in ideas about chaos, and it is an excellent starting point for our analysis that will be related, obviously, to complexity, given that it means diversity, creativeness, interactions at different levels and the possibility of emergent properties.

The paradigm of the complexity constitutes an obligatory reference to the scientific analysis on the verge of the XXI century, and this is especially true in the field of Economics, which is accustomed to navigate against the tide, and subject to a continuous dispersion, such as the one we had the occasion to emphasize in a paper we published some years ago (Fernández Díaz, 2000, pp.39-44).The appeal of the complexity, in the case of the Science of Economics, warrants its interpretation as a healthy exercise, worthy of defence before the reductionist determinism professed by our science throughout its history, succumbing all too often to the aesthetic pleasure of simplicity. Should it not be clear, and with a view to taking a stand, it should be stated, with no undue delay, that determinism is pernicious: when we are distracted we deliberately skip the complexity.

We obviously do not seek the disqualification of determinism in radical support of decisive indeterminism, nor enter into the protracted and inconclusive polemics set forth all around it. What really and powerfully attracts the attention is the fact that a natural science, such as Physics, may have spectacularly advanced, which supposes its quantum revolution, through basis on the Heisenberg's uncertainty principle and the well known interpretation of Copenhagen, due to Niels Bohr, in as much as, it reigns over a rigid determinism in the most complex sphere of the spirit, of the culture and society, or, tantamount to the same, of a social science such as Economics.

With a view to entrenching the question momentarily, and without hazard of remission to other more specific and detailed works on the matter, determinism could be understood as an abstraction and simplification, to make everyday complexity intelligible, considering, for its part, indeterminism as a consequence of our inability to explain complication, due to the fact that we do not avail of sufficient information. From this point of view, indeterminism would then be a clear consequence of complexity.

Below the broad parasol of complexity, we find ourselves in a fascinating world of concepts, terms and instruments, bustling, intertwined, and opening new horizons in almost all fields of knowledge. Dynamics, non-linearity, irregularity, order and chaos are only some of them, behaving as parts of an indivisible whole. Chaos, habitually considered a subset of complexity, constitutes, without doubt, one of the key pieces of the process, and due to this, we can speak within our scientific field of *Chaotic Dynamic Economics*.

But, can Economic Science be set forth and understood in terms of complexity?. Evidently, it can and should be done, especially when dealing with an empirical science situated within the scope or group of *socials*, as we have already seen, and if it is taken into account that complexity is consubstantial and ubiquitous. If we remember the three types of complexity considered by Atlan (1991, pp 9-38), it could be easily proven that Economics exhibits or presents all of them, both quantitative as well as the essentially qualitative type. In effect, besides the probabilistic natural complexity, the most proper and direct, there is the algorithmic complexity, for instance, in the example of computer processed models of equilibrium, such as those carried out by Scarf in his well known work on matter. Likewise, one might undoubtedly speak about complexity in the appreciation of Economics, which, on the other hand, involves recognizing and admitting the existence of a subjective indeterminism in the sense already brought to hand. To all this, it is necessary to add a type of strict definition of complexity, which is habitually used in Economics in the more recent works of specialists. In them, the term *complex* is used, to refer to those cases in which dynamic long-term behaviour is more complicated than a fixed point, a cycle limit, or a torus; or tantamount to the same, when chaotic behaviour is produced.

Important lines or sub-headings of the Economic Analysis and Economic Policy fall full and can be included in the scope of the Economics of Complexity. In these scopes, subjects referent to monetary dynamics or keynesian dynamics are undertaken, problems of inflation and unemployment, the determinants of endogenous cycles, growth and distribution models, the development of exchange rates, the existence of chaos in capital markets, or the non-linearity and chaos in time series. In all of them, one tries to count on new approaches and methods that allow an analysis closer to the truth or reality, and to the intrinsic and inevitably dynamic nature of economic phenomena.

Within the specific framework of financial markets there are many studies that test for non-linear dependence on daily stock indices, or searching for evidence of chaos in the future prices of commodities. It is also very important to search for the implications of non-linear dynamics for reasons of financial risk management; the chaotic behaviour in exchange-rate series, or nonlinearities and chaotic effects in option prices. Finally, non-linear modelling with neural networks offers a very interesting and efficient approach for studying the prediction of chaotic time series (Trippi, 1995, pp. 467-486), and has been utilized successfully in different branches and problems of Economics.

It is necessary to remember that the major problem in time series research is the difficulty of distinguishing between deterministic chaos and a purely random process, taking into consideration that the most important characteristic of chaotic dynamical systems is their short-term predictability. Chaos is at the same time disorder and determinism. Chaos, in principle, due that is apparently disordered, make non predictable its evolution. But, on the other hand, being deterministic, and governed by systems of non-linear equations, it should be possible to predict and control once you know the mathematical relationships of the variables that influence it. As said Henri Poincaré is much better to look farther without having certainty, that don't look anything at all. Because of this we must to undertake the analysis of the main concepts, techniques and mathematics of chaos, that is, of all the weapons we need to know and to deal with an irregular and complex reality, as already we have pointed out.

Before of going on, however, it is necessary to know that complexity and chaos are intimately related to the concept of emergence. What does emergence means?. Taking into account the evolution approach, we can think that emergent evolution may be interpreted as an incessant flow of creative novelty, which implicate a special conception of the whole and the parts, farther away the simple and lineal idea of an additive process (Fernández Díaz, 1999a, pp. 139-143). In reality there is a process of emergence when the behaviour of the overall system cannot be obtained by summing the behaviours of its constituent parts. That is, the whole is indeed more than the sum of its parts.

The habitual definition of chaos, which hallmark is the sensitive dependence on initial conditions, implies that there is no information within chaos, and it has neither form nor structure. For us, chaos may be complex and appear to be non-deterministic, but hidden within it is a wealth of information. If in an emergent phenomenon there is also some hidden information, given that, as we have seen, the whole became something more than the sum of its components, seems clear, first, the closed relation between chaos and emergence, and secondly, the help that the last one can render to the predictability of chaos. We must remember this very important consideration in the concluding remarks of this work.

NEW TECHNIQUES AND THEIR POSSIBILITIES: THEORY OF CATASTROPHES AND MATHEMATICS OF CHAOS

The characteristics of irregularity and non-linearity are, among others, derived from the complexity of economic behaviour, which oblige, as stated at the beginning, the utilization of concepts and new instruments especially conceived to face challenges, which today arise within Economics, and of course, in other fields of knowledge. Amongst them, the Theory of Catastrophes, and very especially, the Mathematics of Chaos stand out.

The majority of authors coincide in as much as the Theory of Catastrophes and the Mathematics of Chaos can be considered as two approaches to a general theory of dynamics of discontinuities. Both have in common as a base the idea of a splitting or halving of the equilibrium at critical points, just as the fact that functional relation are, with greater frequency, of the non-linear type. But they differ in as much as some discontinuities are set forth on a great scale: the Theory of Catastrophes, and others on a small scale: the Theory of Mathematics of Chaos (Barkeley Rosser, 1991, pp. 2-3). The Theory of Catastrophes is therefore a special case of the bifurcation theory accredited originally to Poincaré, which contemplates the world as essentially uniform and stable yet subject to sudden changes, the unexpected, or discontinuities on a grand scale which are produced in certain variables of state.

It is well known that the starting point of the Theory of Catastrophes can be found in the works of René Thom and Christopher Zeeman, at the end of the sixties and the beginning of the seventies. On other occasions we have, and at certain length, taken to hand this new mathematical method, in order to describe the evolution of forms in nature, by hazarding even some economic applications, and concretely, the problem of stagflation[1]. We shall not go into this any further. Instead, we shall center our attention on Chaos and its measurement, which has greater relevancy for the purposes of our analysis.

It is often said that chaos is a ubiquitous phenomenon which is produced everywhere and can be observed in all fields of Science. Thus we find chaotic systems in the Hamiltonians, in the three bodies of celestial mechanics, in the physics of fluids, in lasers, in particle accelerators, in biological systems, in chemical reactions, and as we shall soon see, in no small part of the behavioural forms within the field of Economics.

Chaos can be located through the function of ***strange attractors***, by following bifurcation diagrams, or by analyzing the intricate profile of figures of fractal geometry. It should not be forgotten, in this respect, that, as Giambattista Vico said, chaos is *the raw material of natural things that, shapeless, is thirsty for form, and devours all*. We know that the essential geometrics of chaos consist of stretching and bending, as pointed out by Stephen Smale in his topologic transformations. In effect, the irregularity of movement is produced by a mechanism which is broken down into two actions. On

[1] See Fernández Díaz (1999b), pp. 45-48. For a more complete and up-to-date analysis we recommend: Castrigiano and Hayes (1993).

the one hand, the spatial phase is stretched, by separating trajectories, and then it doubles back onto itself.

The exponents of Lyapunov serve to explain the first part of this process, on giving a measurement on the exponential separation of two adjacent trajectories. We can study local instability of a discrete system $x_{n+1} = f(x_n)$ in the Lyaponov sense measuring how two adjacent points separate with the iterated application of the function, that is,

$$|f^n(\mathrm{x}_0 + \epsilon) - f^n(\mathrm{x}_0)| = \epsilon^{n\lambda(x_0)} \tag{1}$$

The limit

$$\lambda(x_0) = \lim_{n\to\infty} \lim_{\epsilon\to 0} \frac{1}{n} \log \left|\frac{f^n(x_0+\epsilon)-f^n(x_0)}{\epsilon}\right| \tag{2}$$

or also:

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \log \left|\frac{df^n(x_0)}{dx_0}\right| \tag{3}$$

are both (2) and (3) path expressions of the Lyapunov exponent. In a general n-dimensional system there will be *n* Lyapunov exponents, showing each of them the average rate of expansion or contraction of the phase space in each of its *n* direction under the action of the dynamical system.

These Lyapunov exponents can be used to distinguish simple dynamic attractors from complex dynamics attractors, that is, those that we could call existing traditional attractors until the Lorenz contribution in 1963, *fixed points*, *limit cycles* and *quasi-periodic torus,* from chaotic or strange attractors.

We can say that the dynamics inside fixed points, limit cycles or torus is a simple dynamics in the sense that two orbits that start arbitrary near each other, always remain adjacent, thus providing a guaranteed predictability on a long term basis. However, inside the strange or chaotic attractor the dynamical system is complex in the sense that although bounded inside the attractor is also locally instable, with recurrent but aperiodic cycles, and with sensitive dependence to initial conditions making predictions difficult beyond the very short term. Therefore “simple dynamics” and “complex dynamics”, can be detected by means of the Lyapunov exponent. More concretely, a positive Lyapunov exponent, i.e. local instability, is a necessary condition for the existence of chaos.

If we are situated in a dimension, we have only stable fixed points with the exponent λ negative. In the case of two dimensions, the attractors would be fixed points, with negative exponents, and limit cycles with (λ_1, λ_2)=(-,0).

In three dimensions one would have:

$(\lambda_1, \lambda_2, \lambda_3) = (-,-,-) \rightarrow$ (stable fixed point)

$(\lambda_1, \lambda_2, \lambda_3) = (0,-,-) \rightarrow$ (stable limit cycles))

$(\lambda_1, \lambda_2, \lambda_3) = (-,0,0) \rightarrow$ (stable torus)

$(\lambda_1, \lambda_2, \lambda_3) = (+,0,-) \rightarrow$ (strange attractor)

There are many examples of strange attractors in specialized literature, beginning with Stephen Smale, who supplies through a topological transformation a base for the understanding of chaotic properties of dynamic systems. Among the stated attractors, the Lorenz attractor stands out, which takes on the form known as the wings of the butterfly, and in the one that borders on an important problem for meteorology, the one about atmospheric convection, consisting of the evolution of a layer of fluid heated from below. To this emblematic attractor, other examples should be added, such as the not less known one of Rössler, and the one by Hénon, which has an elegant structure and is quite complex[2]. But we cannot speak of strange attractors without entering into the attractive and fascinating field of fractals.

The *chaotic attractors* are fractals. Fractals are geometric objects which have a beautiful microscopic structure that have been developed within the framework of a new form of the geometry of nature or of the complexity created by Benoît Mandelbrot, who in 1975 published his famous work called ***The Fractal Geometry of Nature***. He was educated at "Ecole Normale" and the "Normale Polytechnique" and, with his original formulation, intended to confront the unbounded formality of the Bourbaki group, thereby reestablishing the image and prestige of Henri Poincaré.

The fractal concept involves a new idea of dimension beyond the Euclidian one, given that it dealt with the fractal or intermediate dimensions that come, in essence, into prominence if it is considered that the system does not occupy all the space that corresponds to its Euclidean dimension. In effect, the fact that the dimension may be inferior to the number of parameters or degrees of freedom, necessary to completely specify the state of the system considered, signifies that it does neither exploit all the possibilities, nor all the states theoretically possible.

The fractal should be understood as a geometric form which remains unaltered, whatever the increase in which it is observed. It could be said that, within reasonable, the fractal has the same structure on all scales, the contrary to what occurs in the phenomenon of the renormalization in which the figures notably alter when they are modified or vary.

The fundamental problem of fractals lies in knowing their dimension, which does not necessarily need to be an integer, as we have already noted. Originally, the numeric measurement of the degree of rigorousness was denominated as the Hausdorff-Besicovitch dimension; today it is called the fractal dimension. In general terms, the dimension is a measure of the occupation of space by a geometric object. Normal non-

[2] For a detailed study, see: Thompson and Stewart (1991, pp. 212-253) and Fernández Díaz (1994, chapter 6).

fractal objects have a Hausdorff dimension equal to its topological dimension. However, fractals have a Hausdorff dimension strictly greater than its topological dimension

One of the methods employed for carrying out the measurement of fractals is based on the concept of homotecia in Euclidian geometry, which also allows the calculation of fractal dimension through basis on the concept of capacity:

$$D_0(S) = \lim_{\epsilon \to 0} \frac{\log M(\epsilon)}{\log\left(\frac{1}{n}\right)} \tag{4}$$

where S is a subset of the n-dimensional space, and $M(\varepsilon)$ the minimum number of ε-side n-dimensional cubes necessary for covering such a subset. For small values of ε, the implicit definition shown in (4) means that:

$$M(\epsilon) \propto K \cdot \epsilon^{-D_0} \tag{5}$$

The capacity of a point, a line or an area in the bi-dimensional space, takes the values 0, 1 and 2 respectively. That is, if we take the cubes of the side ε, the number required to cover the point would be proportional to $1/\varepsilon^0$, to cover the line to $1/\varepsilon^1$, and to cover the surface to $1/\varepsilon^2$. The dimension of fractal sets, as mentioned previously, is strictly greater that this Euclidean capacity. Thus, the Kotch curve and the Cantor set, which constitute typical examples of fractals, have fractal dimension greater that one and 0, respectively:

$$d = \frac{\log 4}{\log 3} = 1.2619 \qquad and \qquad d = \frac{\log 2}{\log 3} = 0.6309 \tag{6}$$

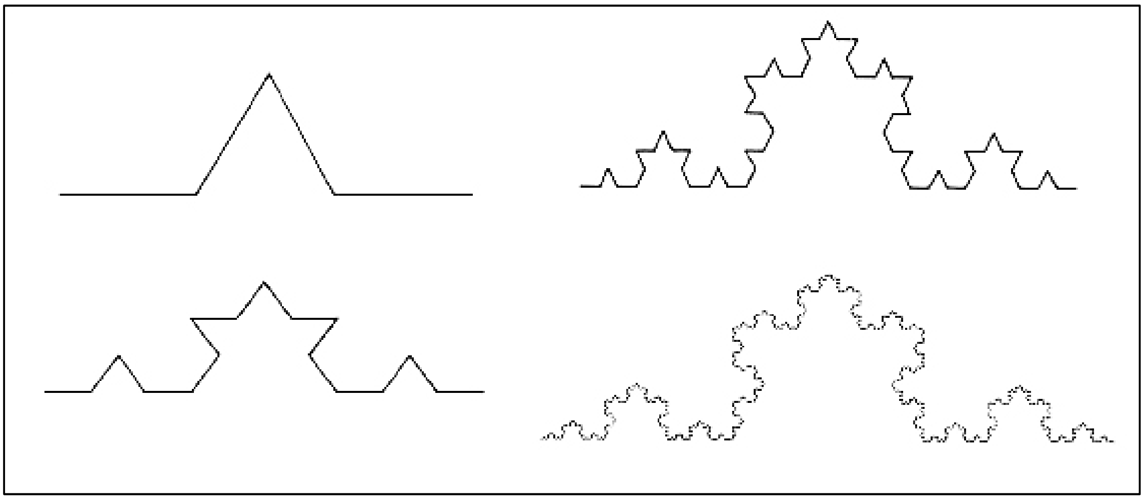

Figure 1. Kotch curve

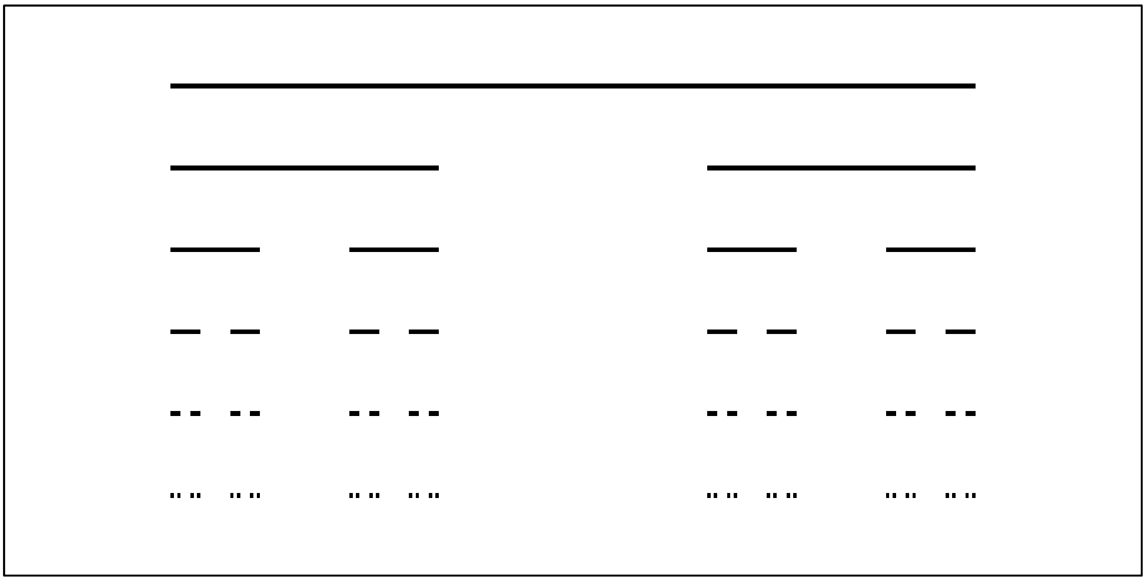

Figure 2. Cantor Set

Lyapunov exponents and fractal dimension are the two main instruments of the Mathematics of Chaos. Another key concept in the analysis of chaos and complex dynamics is bifurcation, which could be defined as a doubling of the period of the attractors as some parameter of the system changes. With the analysis of the occurrence of this bifurcation, that is, the comparative dynamics of the system when the value of a parameter is changed, it is possible to explore the different types of attractors that the model can achieve. And the way in which the changes in the parameters move the system from one attractor to another is precisely through the bifurcation of the period.

Without forgetting the contributions of Yorke and May, and of course, the clear and decisive inspiration of Smale, the best known and illustrative analysis of comparative dynamics is the *Feigenbaum bifurcation tree*. Consider, for example, the logistic map:

$$x_{t+1} = \mu x_t(1 - x_t) \tag{7}$$

where μ is a constant situated at the interval [0,4]. When we iterate the map from an arbitrary initial condition, the attractor of this discrete dynamic system is obtained depending on the values assigned to μ:

If	$0 \leq \mu < 3$	→	a sole stable fixed point
	$\mu = 3$	→	a marginally stable fixed point
	$\mu > 3$	→	a fixed point becomes unstable
	$\mu = 3{,}2$	→	second period cycle
	$\mu = 3{,}5$	→	fourth period cycle
	$\mu = 3{,}56$	→	the period has doubled to eight
	$\mu = 3{,}567$	→	the period has doubled to sixteen
	$\mu = 3{,}58$	→	the cascade of duplications is so rapid that the logistic map becomes chaotic.

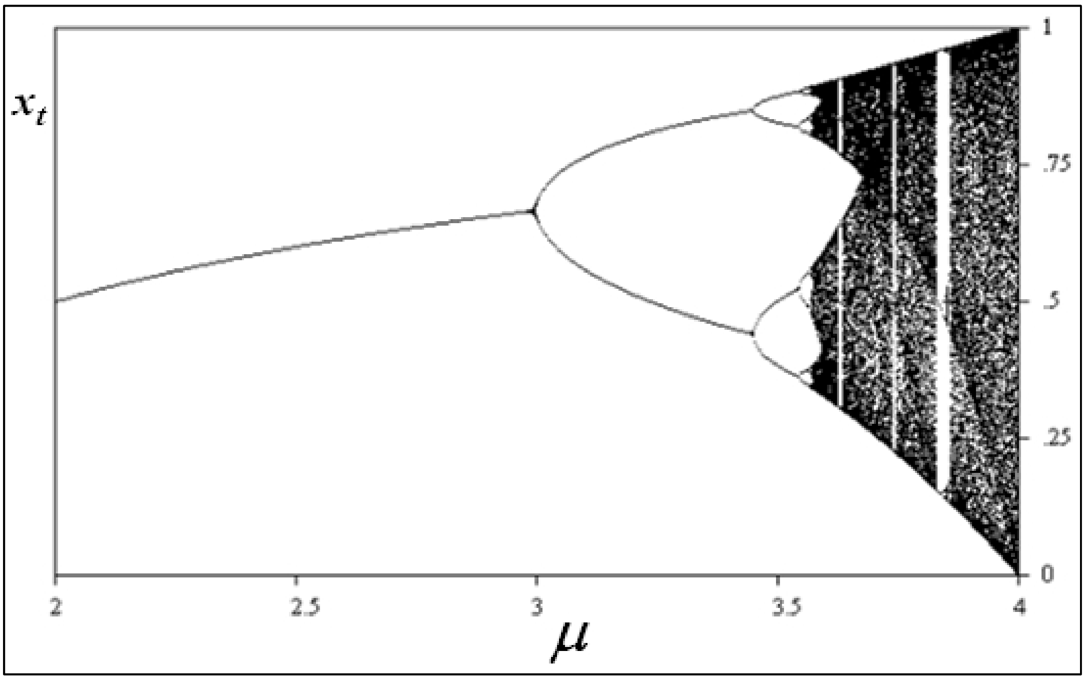

Figure 3. Feigenbaum Tree

In this way, by advancing through the duplication or the bifurcation of the period of the attractor, the Feigenbaum Tree is obtained (*figure 3*).The scale factor of the tree branches tends towards a universal Feigenbaum constant, which is equal to 4.6692, and is maintained even if we use another application. In effect, the relation

$$\delta_i = \frac{\mu_i - \mu_{i+1}}{\mu_{i+1} - \mu_{i+2}} \tag{8}$$

converges towards this universal number, being therefore $\delta_\infty = 4.66922$.

It should be added that, in the zone of chaos, small windows or oases of order and stability can be found in the middle of disorder, illustrated in the habitual graphic representations by means of clear spots in foggy and obscure areas.

There are abundant contributions in the literature showing that it is possible to generalize the traditional theoretical economic models to show chaotic behaviour under economically plausible assumptions.[3] However, while there is not a great difficulty to design theoretical models in regime of chaotic behaviour, there is no clear evidence that economic time series behave chaotically.

In fact, the major advances in the application of chaos theory in Economics deal with the tools to detect if the underlying true economic time series generation process is really a chaotic dynamic system. The detection of chaos in the underlying dynamics of a time series is divided into several stages. The first step in detecting chaotic behaviour from a time series is to find evidence of non-linear time dependence in the underlying dynamics of the system. And for that, the most widely used test in the field of economics are the BDS test (Brock et al. 1996) and the Hurst exponent (Mandelbrot,

[3] See Fernández Díaz, A. (2000).

1971,1972 and 1997), which are two techniques to contrast the existence of time dependence, linear or non-linear.

To detect non-linear dependence the Brock (1986) test is used. This test consists on filtering the time series by a general auto-regression model with a range large enough to ensure that any linear dependence has been completely removed. If, despite the linear filtering, Hurst and BDS tests continue to show evidence of time dependence, then it must be non-linear.

Once detected the non-linear dependence, the next step is to estimate both the Lyapunov exponent and the fractal dimension of the attractor of the underlying system generating the time series in order to test if that dynamical system presents chaotic behaviour. The main limitation of this approach is that this system is unknown. It is for this reason that a previous step (prior to the estimation of Lyapunov exponents and fractal dimension) is the recovery or reconstruction of the attractor but maintaining the qualitative properties of the underlying unknown dynamical system generating the time series.

A commonly used method of reconstruction of the attractor is the *lag method*. This method is based on the *embedding* theorem of Takens (1985), that establishes that, under certain conditions, though it will not be possible to reconstruct the orbit of the dynamical system in the original phases space, it is possible to obtain an approximation of it that result equivalent in a topological sense (equivalence in the dynamic and geometric properties), and that permit to extract all the relevant information about the unknown underlying dynamical system that generates the time series.

To show such a reconstruction technique, we suppose a process generated by a n-dimensional dynamical system defined in continuous time,

$$\dot{x_t} = f(x_t) \ ; \ x \in D \subset \mathbb{R}^n \ ; \ t \in \mathbb{R} \tag{9}$$

or in discrete time:

$$x_{t+1} = f(x_t) \ ; \ x \in D \subset \mathbb{R}^n \ ; \ t \in \mathbb{N} \tag{10}$$

We suppose that (9-10) is an observable process, but that the observer does not know neither the structural form of the dynamical system (9-10), neither its dimension n, nor the exact value of any state variables. What is only known is a scalar signal that the observer can measure (y_t). In general, the scalar signal will be a function h –also unknown for the observer– of the n state variables of the dynamical system:

$$y_t = h(x_t) \tag{11}$$

In this way, the observation in time of scalar signal y_t provides a sample or time series of size N that we have to use to extract information on the qualitative properties of the unknown original dynamical system. That is to say, what is intended with the technique of the reconstruction of the attractor is to reconstruct or to extract information on the hidden dynamics of $\boldsymbol{f}$ using the observed time series y_t . In fact, as the vector field $\boldsymbol{f}$ as well as the function h are unknown, the n-dimensional original phases space cannot be reconstructed. However we can approximate it in a pseudo-phases space –embedding space– that reproduces the original dynamics.

The lag reconstruction method is based on the projection of the scalar signal in others *m*-dimensional vector variables $\boldsymbol{y}_t$ *–embeddings–* that will be built using the delayed values of the time series y_t: t=1,2, ... , N:

$$\boldsymbol{y}_t = \left(y_t\,, y_{t+\tau}\,, y_{t+2\tau}\,, y_{t+3\tau}\,, \cdots\,,\ y_{t+(m-1)\tau}\right) \tag{12}$$

being τ the reconstruction lag (a fixed sampling period between two consecutives observations) and *m* the embedding dimension. In this way, it is possible to obtain a *m*-dimensional path of size $N\text{-}(m\text{-}1)\tau$:

$$\begin{aligned}
\boldsymbol{y_1} &= \left(y_1\,, y_{1+\tau}\,, y_{1+2\tau}\,, y_{1+3\tau}\,, \cdots\,,\ y_{1+(m-1)\tau}\right)\\
\boldsymbol{y_2} &= \left(y_2\,, y_{2+\tau}\,, y_{2+2\tau}\,, y_{2+3\tau}\,, \cdots\,,\ y_{2+(m-1)\tau}\right)\\
\boldsymbol{y_3} &= \left(y_3\,, y_{3+\tau}\,, y_{3+2\tau}\,, y_{3+3\tau}\,, \cdots\,,\ y_{3+(m-1)\tau}\right)\\
&\ \ \cdots\\
\boldsymbol{y_{N-(m-1)\tau}} &= \left(y_{N-(m-1)\tau}\,, y_{N-(m-1)\tau+\tau}\,, y_{N-(m-1)\tau+2\tau}\,, \cdots\,,\ y_N\right)
\end{aligned} \tag{13}$$

When τ and *m* are adequately chosen, then the orbit described by (13) can be used to approximate the n-dimensional orbit described by the original system.

Concerning the election of the optimum lag τ, we can indicate that it would be neither very small nor very large –figures 4-9–. When τ is too small, it will not have passed sufficient time between the two observations, and then the evolution of the embeddings will not provide new information on the original phases state, this is, the values of y_t and $y_{t+\tau}$ will be identical to practical effects, linearly dependent or not sufficiently independent as to distinguish the information that they are capable of revealing on the dynamics of the underlying system. On the other hand, the lag τ would not be very large, since then, y_t and $y_{t+\tau}$ will be very apart in time and if the dynamical system is chaotic and presents sensitive dependence with respect to the initial conditions, it will not be possible to detected the correlation or dynamic dependency between both values. That is to say, it will not be able to assure that both values are connected in the time by the dynamical system (9- 10). In this case, if the system is chaotic, the geometric figure drawn by the reconstruction of the attractor would be confused with a purely stochastic uncorrelated process.

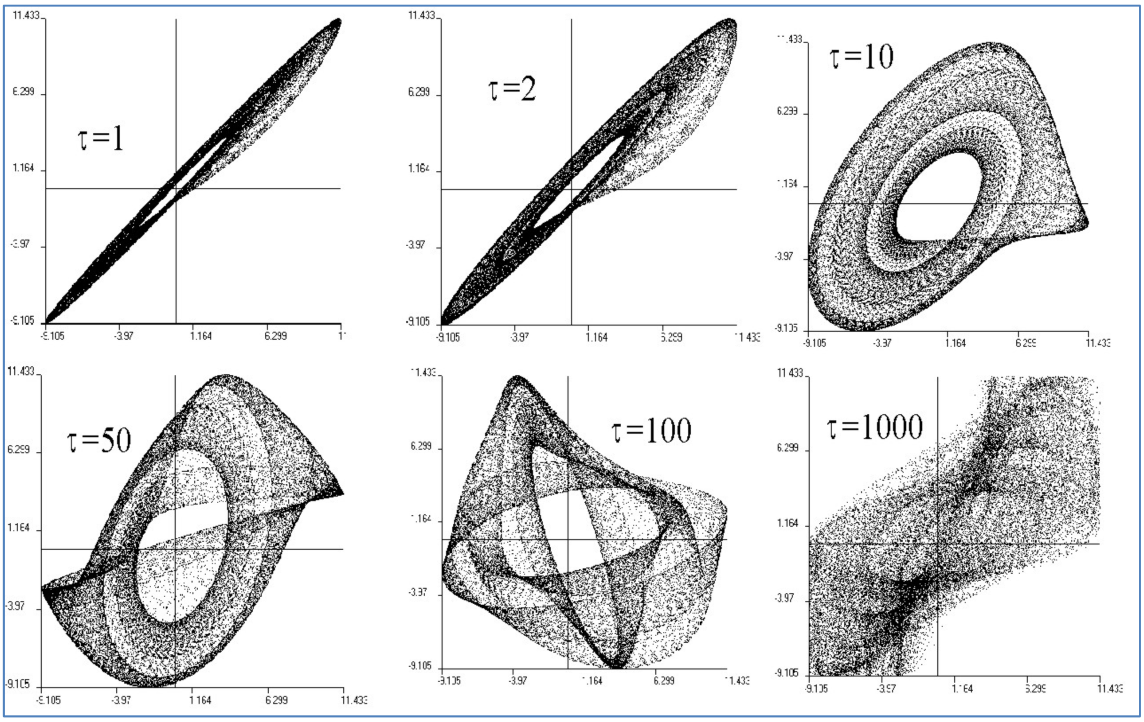

Figure 4. Reconstruction of the Rössler chaotic attractor, projections on the plane for different lags of x_t

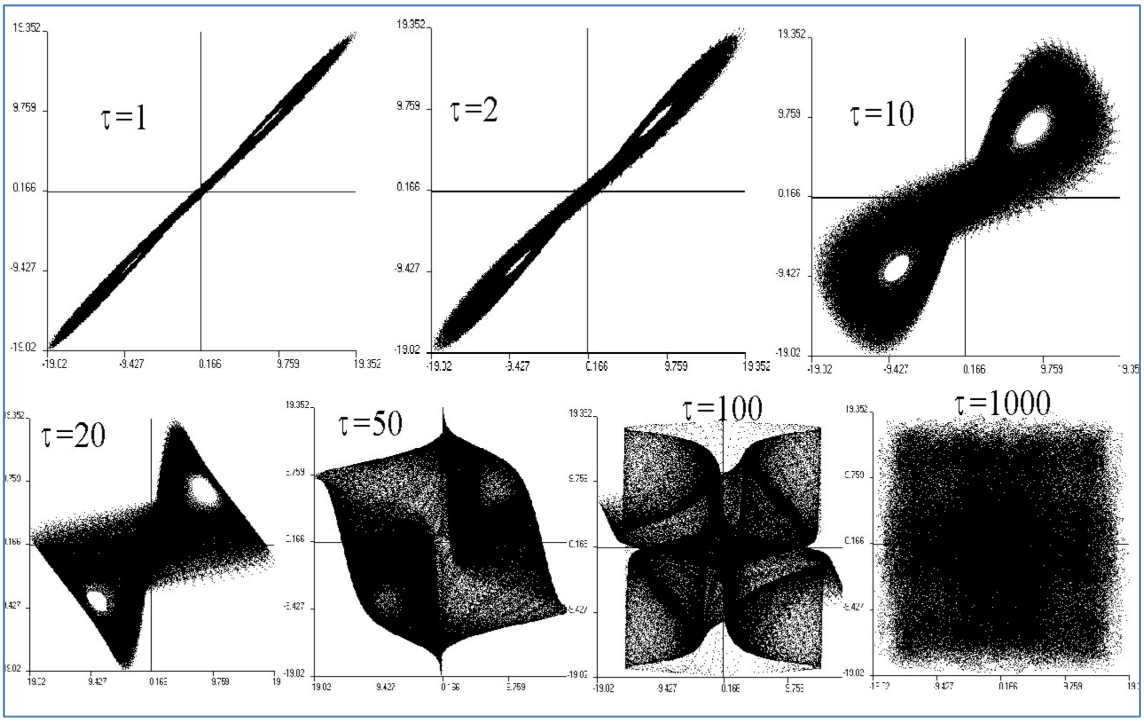

Figure 5. Reconstruction of the Lorenz choatic attractor, projections on the plane for different lags of x_t

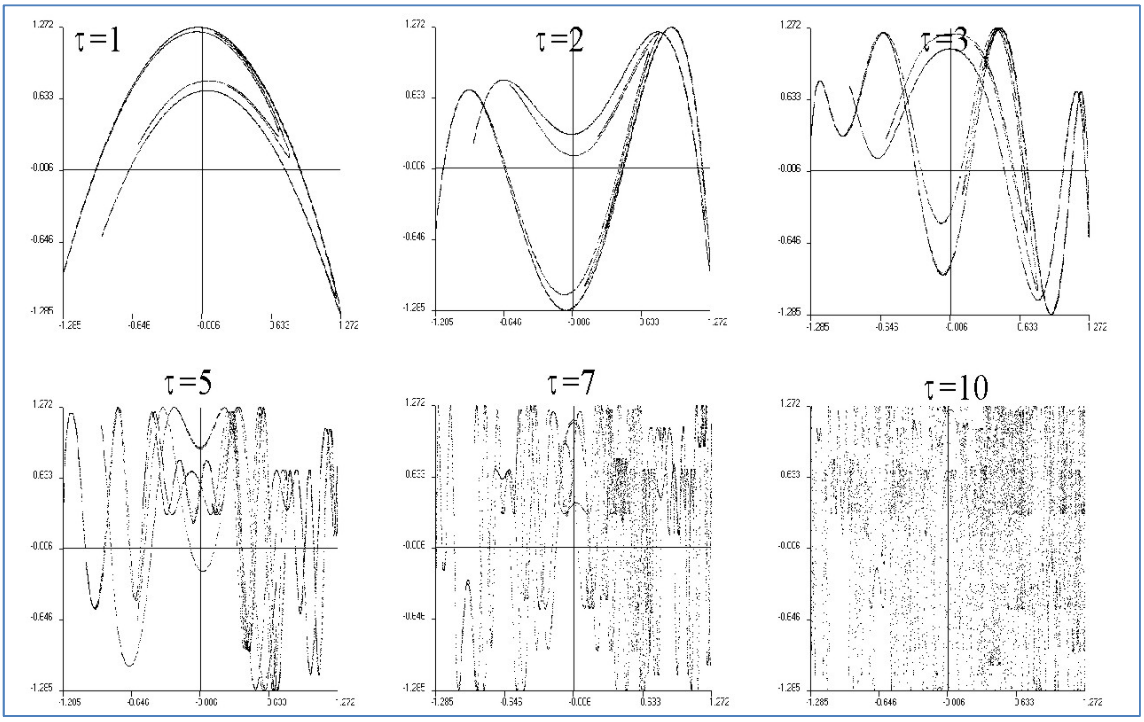

Figure 6. Reconstruction of the Hénon choatic attractor, projections on the plane for different lags of x_t

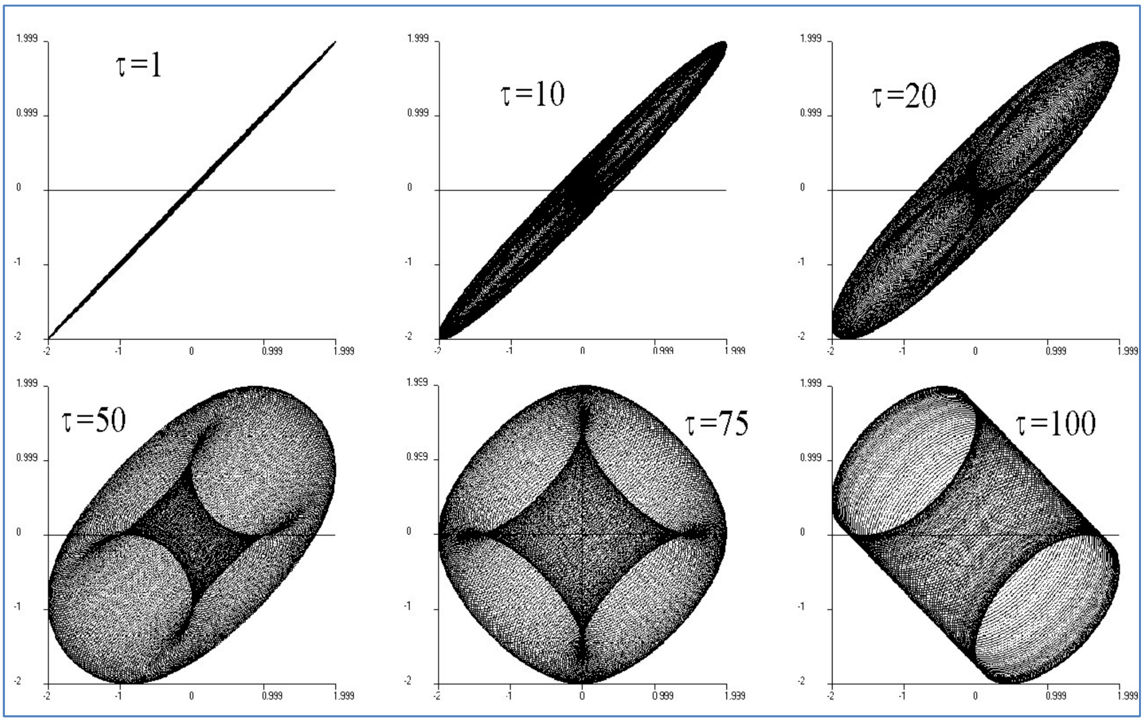

Figure 7. Reconstruction of a toroidal cuasi-periodic attractor, projections on the plane for different lags of x_t

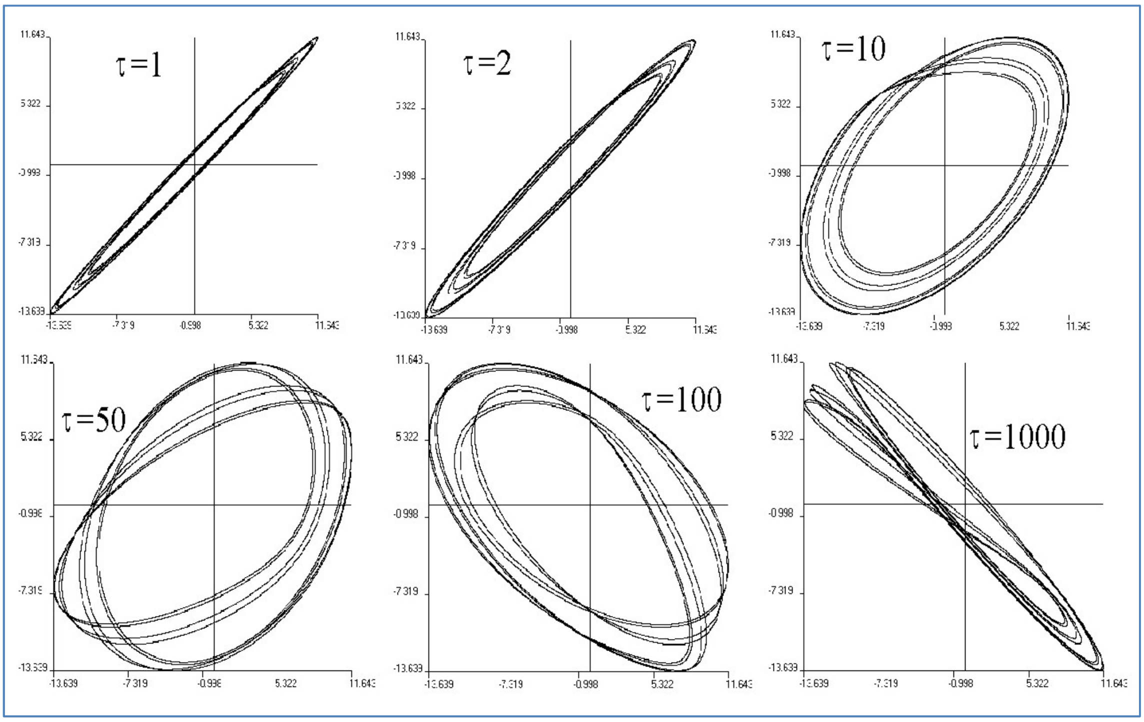

Figure 8. Reconstruction of a limit cycle period eight (from the Rössler model), projections on the plane for different lags of x_t

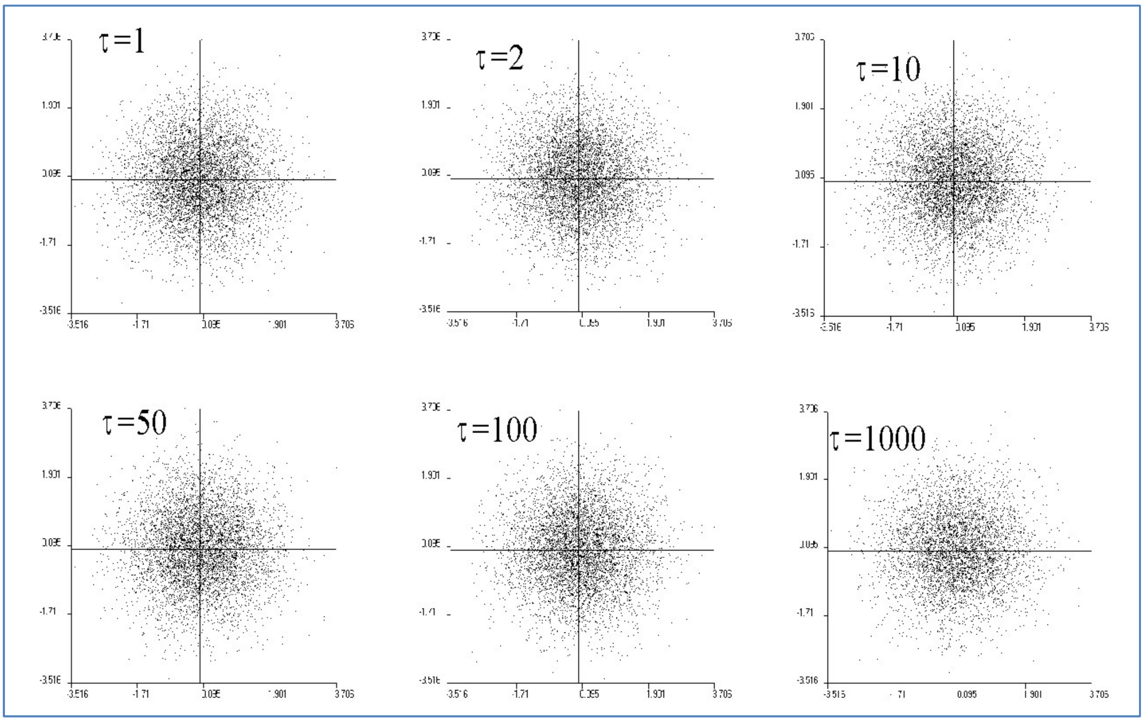

Figure 9. Reconstruction of a IID random walk N(0,1), projections on the plane for different lags of x_t

A method for the election of the optimum lag in the reconstruction of the attractor was proposed by Fraser and Swinney (1986) who use a general independence criterion based on the average mutual information function. This function supposes a generalization of the linear correlation function and it is capable of capturing any type of correlation, linear or non-linear, between two observations separated by a lag τ. The mutual information function measures the information that can be obtained on $y_{t+\tau}$ from y_t:

$$I(\tau) = \sum_{t=1}^{N-t} p\,(y_t\,, y_{t+\tau})\ \log\left(\frac{p(y_t\,,y_{t+\tau})}{P(y_t)p(y_{t+\tau})}\right) \qquad (14)$$

Thus, Fraser and Swinney (1986) proposes to use as lag for the reconstruction the first minimum of the mutual information function, since then, the lag (τ) will be sufficiently high –the two values do not are providing the same information–, but without losing all the information about the dependency provided by the underlying dynamical system –since from the first minimum the mutual information beginning to increase–. This method provides, in general, better results that other methods –for example the first zero of the autocorrelation function–, though it does not exist any formal demonstration that justify its use (figures 10 to 17), and, besides, does not provide information on the chaotic properties of the time series.

After choosing the optimum lag (τ), the next step in the reconstruction of the attractor from the observed time series is the choice of appropriate embedding dimension m. The embedding theorem provides only an inferior limit to the dimension m that must be used in the reconstruction of the attractor, $m>2n$.

There are several contributions that try to detect what is the m required to capture the true dynamics underlying a time series, so as to optimize the information available. One of them is the so-called false neighbours method (Kennnel, Brown and Abarbanel, 1992), which is based on two properties. First, two points in phase space may be neighbours, true neighbours, because the orbit described by the system locates them close to each other after a certain time. Secondly, two points can be neighbours, false neighbours, as a result of the projection of the n-dimensional orbit into a scalar signal and not by the dynamics of the system. Unlike the true neighbours who remain close to each other although they were projected into greater embedding dimensions, false neighbours are no longer close when they are projected into larger spaces. The method of false neighbours consists, precisely, in project the time series into embeddings with increasing dimension, until reach that embedding dimension m for which all false neighbours disappear.

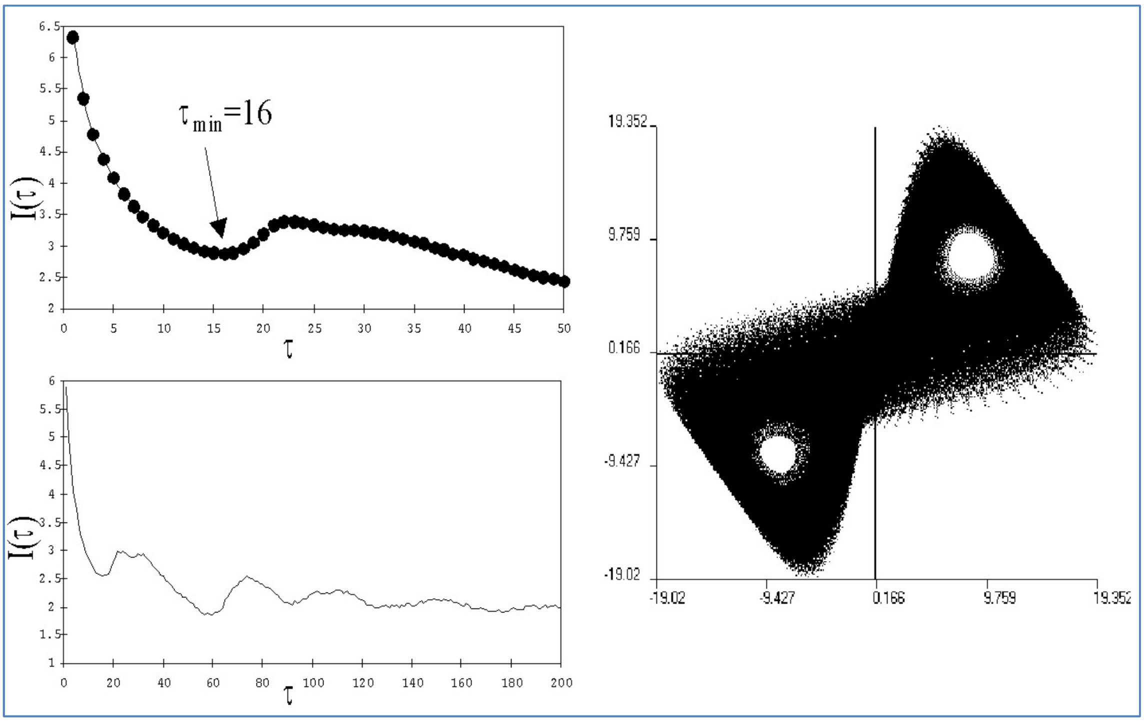

Figure 10 7. Average mutual information, Lorenz attractor

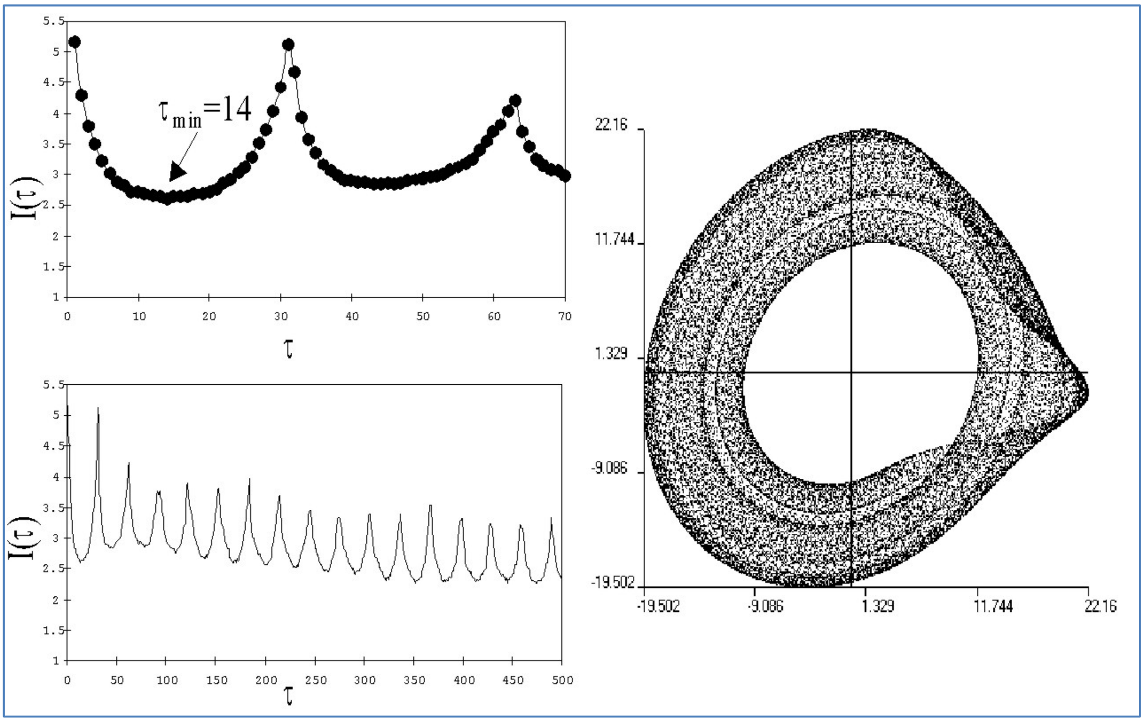

Figure 11 . Average mutual information, Rösler attractor.

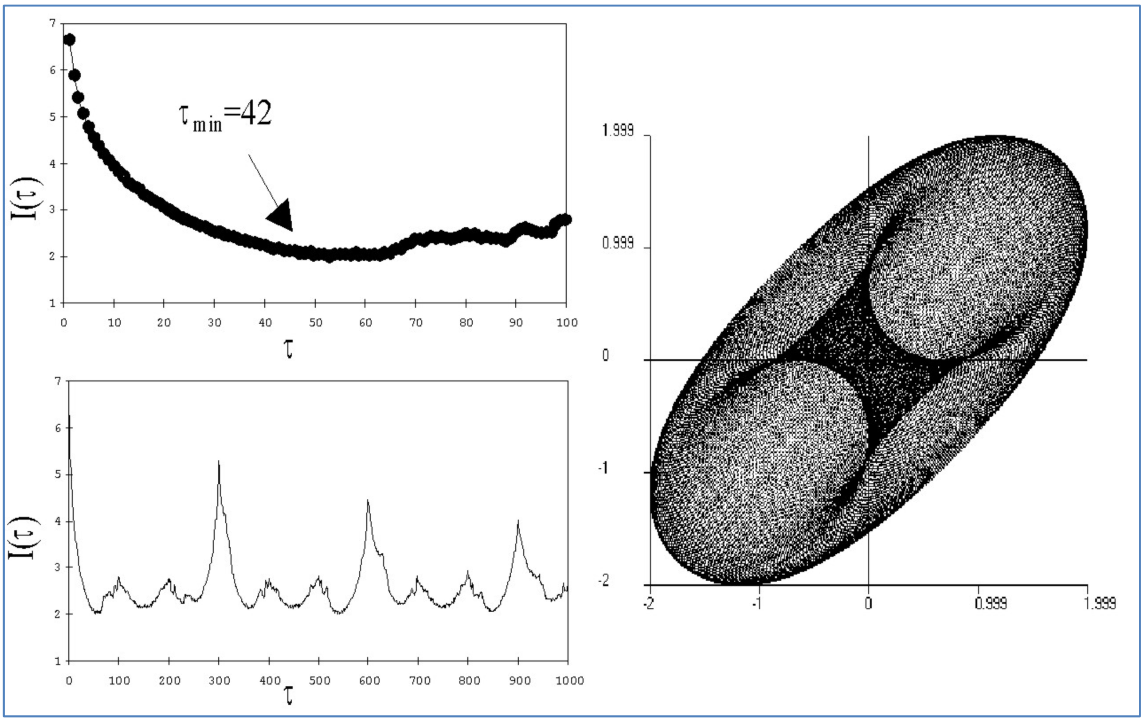

Figure 12. Mutual information, toroidal quasi-periodic system.

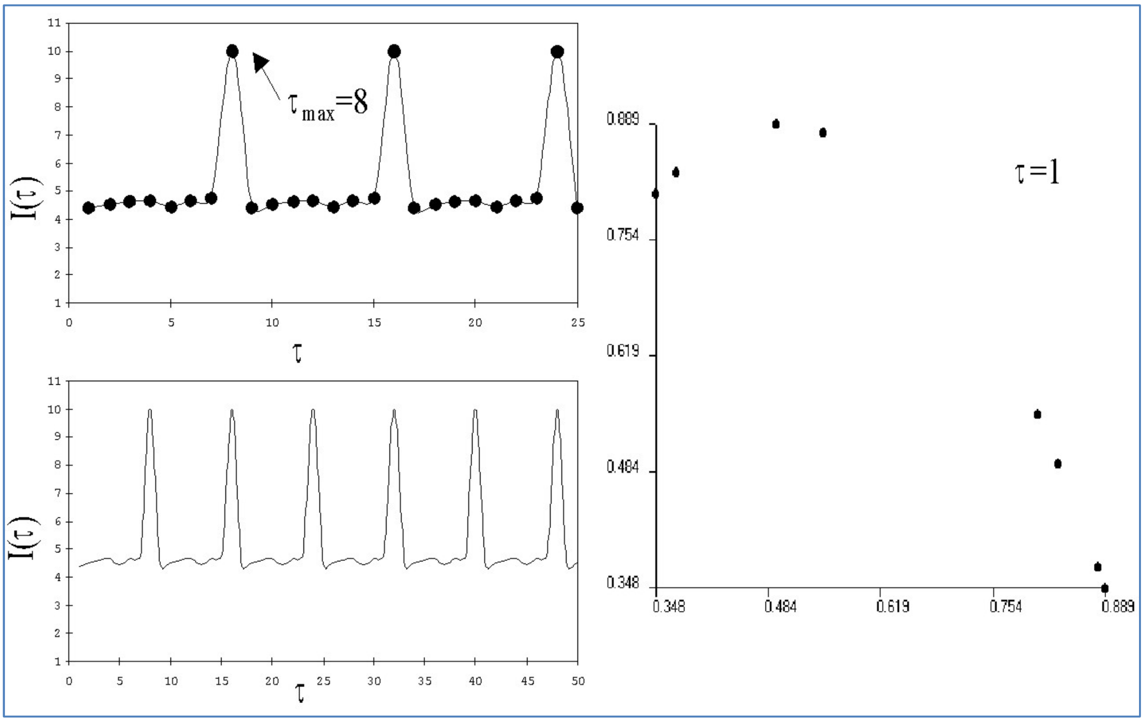

Figure 13. Average mutual information, logistic map μ=3,56-eight period limit cycle.

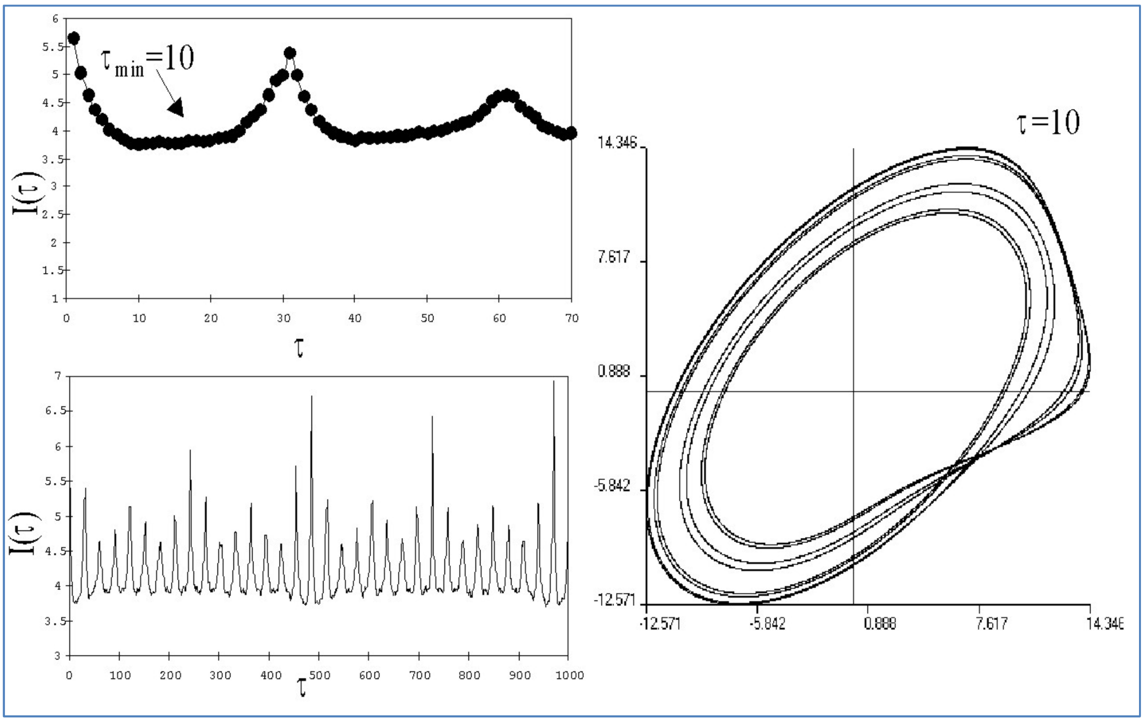

Figure 14. Average mutual information, limite cycle (Rössler model)

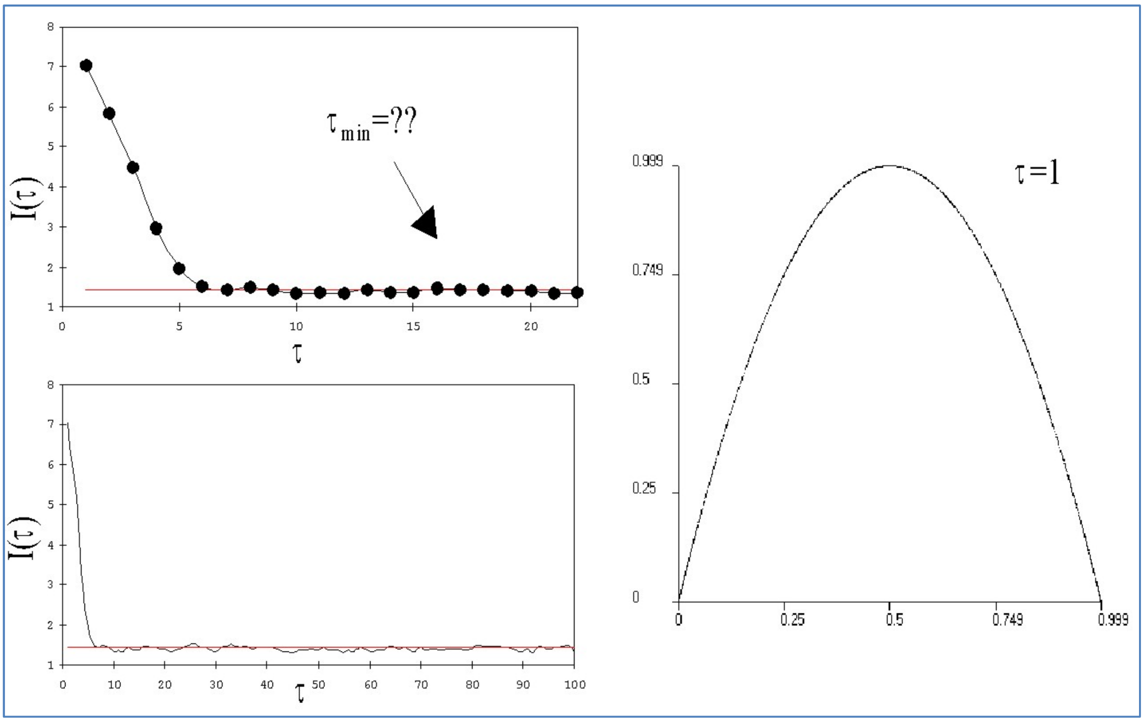

Figure 15. Average mutual information, logistic map μ=4

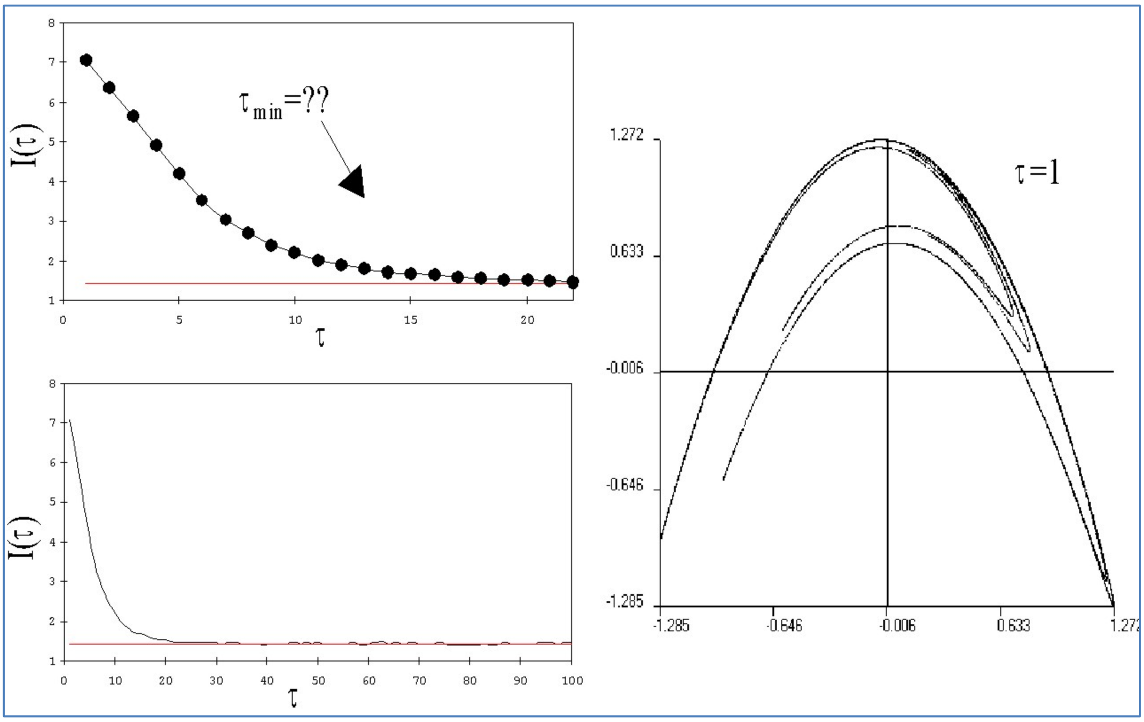

Figure 16. Average mutual information. Hénon chaotic attractor.

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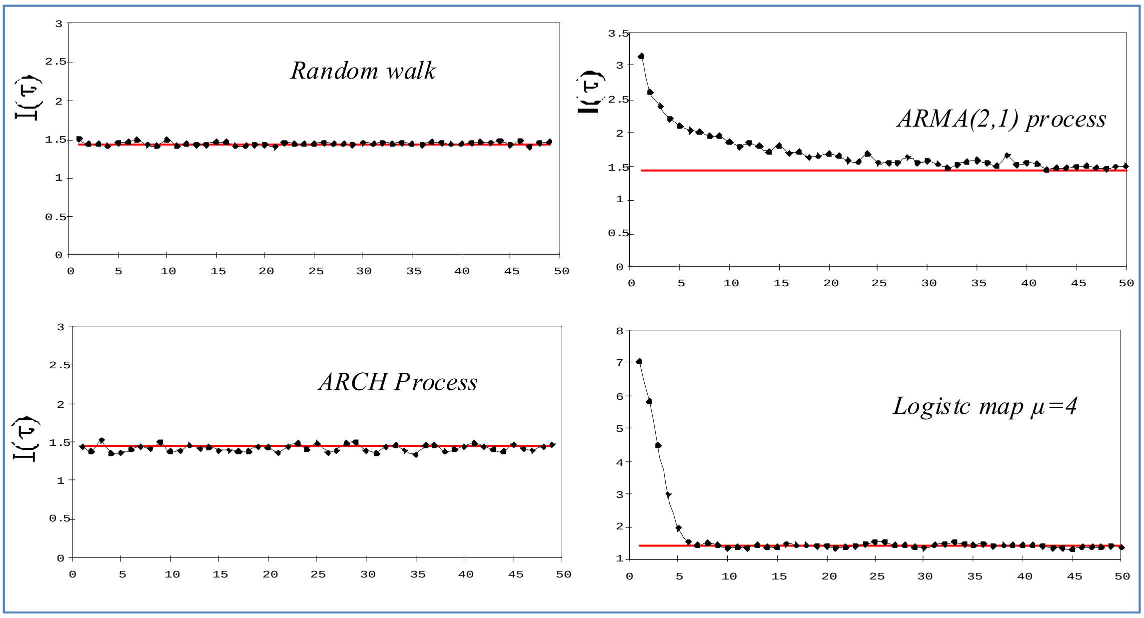

Figure17. Average mutual information, stochastic processe:, ramdom walk, GARCH, ARMA (2,1) and chaotic determinist.

Once we have reconstructed the attractor from the time series, we can proceed now to estimate the fractal dimension and Lyapunov exponents to detect chaotic behaviour.

The Fractal dimension has a metric character, but using alternative measurement concepts to the traditional length, area or volume, and is usually calculated using covering formed by hyper-cubes or boxes. This method for calculating the fractal dimension is, however, little operational when working with embedding dimensions higher than two and when using time series contaminated by purely random noise, and for that reason alternative methods have been developed. Among them are the methods that use the ergodic theory to calculate a probabilistic measure of the attractor, the frequency with which the orbit visits the different parts of the attractor. Among these probabilistic dimensions, the more generally used in Economics is the correlation dimension (Grassberg y Procaccia 1983).

The fractal dimension provides a measure of the complexity of the attractor. However, the estimation of fractal dimension from a time series cannot be taken as a sufficient test for the detection of deterministic chaos. This is because firstly, it is only possible to obtain rough estimates of the true fractal dimension of an unknown dynamic system, and therefore, it is very risky to assure when this approximation is an integer or fractional. Second, because when working with economic time series, we must accept the fact that in the series there is always some random component, so that the estimate of the fractal dimension will be always biased upwards, making it difficult the detection of low-dimensional chaotic behaviour. Therefore, the estimation of fractal dimension in the search for a non-integer dimension and not very high, must be taken as a supplement to other techniques for the detection of chaos, especially, the spectrum of Lyapunov exponents. Recall that in dissipative systems, the presence of a positive Lyapunov exponent is indicative of sensitive dependence to initial conditions, and then it is the sufficient condition to chaotic dynamics to exist

There are several algorithms to measure the Lyapunov exponents of the underlaying time series generation process. The *Wolf et al (1985) direct algorithm* may not provide a correct characterization of Lyapunov exponents of a time series with limited number of observations. Furthermore, the performance of this direct algorithm is very sensitive to the degree of noise in the data. For these reason in Economics, with time series characterized by short sample and error measured, we use indirect methods that use regressions method to estimate the underlaying derivative en (4).

These regression methods to estimate indirectly Lyapunov exponents assume the existence of an unobserved dynamic model that may be chaotic (Gencay and Dechert, 1992):

$$x_t = f(x_{t-1}, x_{t-2}, \dots, x_{t-d}) + e_t \tag{15}$$

where t=1, 2, ..., N ; and $\{e_t\}$ a sequence of *iid* random variables. This model may be expressed in terms of a state vector $\vec{\boldsymbol{X}}_{\boldsymbol{t}} = (x_t, x_{t-1}, x_{t-2}, \dots x_{t-d-1})'$, an error vector $\boldsymbol{\varepsilon}_t$, and a function F: $\mathbb{R}^d \rightarrow \mathbb{R}^d$ such that

$$\vec{\boldsymbol{X}}_t = \boldsymbol{F}(\vec{\boldsymbol{X}}_{t-1}) + \boldsymbol{\varepsilon}_t \tag{16}$$

It is then possible to estimate Lyapunov exponents from the Jacobians of the map, that is, based on non-linear regression estimates of f and F in the respective equations. There are different methods for estimating the map, but the NEGM[4] and the NETLE[5] methods, that take advantage of the use of multilayer feed-forward neural networks models, has been revealed as especially adequate for use with noisy data (Mccaffrey et al., 1992, pp. 682-695) as well as with limited number of observations (Gencay and Dechert, 1992, pp. 142-157). Shintani and Linton (2004) propose some alternatives to estimate the standard error of the Lyapunov exponents estimates with these methods and thus make the contrast of the null hypothesis of almost one LE is positive.

We know very well the special prominence of time series in Economics, and because of this appears convenient to add some complimentary considerations on them, by highlighting those aspects which are related to the non-linearity of chaos. It seems that this specific chapter of Economics had advanced notably, by passing from the use of classic methods, like the moving average, the adjustment of trend, the analysis of decomposition, multiple regression or auto-regressive models, and other more recent ones, such as spectral analysis, adaptive filters, the exponential smooth or Box-Jenkins analysis. But when, in dealing with some problems these instruments do not turn out to be sufficient nor adequate, generalization of the approach are attempted, based on the concept of random walk, which is what happens with martingales or processes of Itô, from a theoretical point of view, or with models such as ARCH, GARCH, EGARCH, and TAR, from an empiric point of view.

In the last years a greater deal of attention has been paid to the possibilities of martingale theory in the field of financial markets. It is also necessary to emphasize the more and more use of stochastic partial differential equations such as the following:

$$\frac{\partial W}{\partial t} = rW - rS\frac{\partial W}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} \tag{17}$$

that is, the well known Black-Sholes, in fact, a Ito's process, where W is the value of a derivative product which underlying security price is S, a random variable that follows a Wiener process, r denotes the risk-free rate of interest, and σ is the annualized standard deviation of *d*S. In the context of the multidimensional Black-Scholes model with market imperfections are used relevant techniques, as backward stochastic differential equations and stochastic optimal control.[6]

CHAOS SEARCH AND ITS APPLICATIONS

The works dedicated to the study of chaos in economics can be divided into two types: the merely empirical model and those looking for modelling economic behaviour, both at micro and macro level, using models that under certain parameter values can lead to know as works the system. Successively we shall undertake six outstanding points or sections: Chaos in time series, Power laws in Economics, Chaotic modelling in

[4] This method has been proposed by Nychka et al. (1992).

[5]This method has been proposed by Gencay and Dechert (1992), Kuan and Liu (1995) and Kuan, CM , Liu T., and Gencay R., (1995, 1997, 2007).

[6] See Nefcti (1996, pp. 101-126), Watsham and Parramore (1997, pp 334-343), Musuela and Rutkowski (1998, pp. 109-134),and Fernández Díaz (1999, pp. 51-59).

Economics, Artificial neural networks applied to Economics, the Fractal road to economic crisis, and Controlling chaos.[7]

Chaos in time series

In the economic analysis we work very often with time series, and because of this is very important to know the possibilities of applying the techniques to study the chaotic behaviour that have been considered previously. All the studies realized about this subject conclude that although evidence of chaos is not definitive, it does detect nonlinearities in macroeconomic series. Some of the test to detect non-linearity and statistical independence and which have proven to be robust to be able to detect nonlinearities that go unnoticed by other traditional test has come from studies of chaos. Prominent among them the BDS (Brock et al., 1996), a test of statistical independence that once cleaned the dependence of the data can be used as a test of non-linearity. Other tests of non-linearity offer also good results, as the developed by Tsay (1986), or the Hinich bispectral approach(1982), with applications to macroeconomic variables by Ashley and Patterson (1989) and Barnett and Hinich (1993).

Although the literature on non-linear dynamics in economics is huge, some of the following articles and books allow acquiring an impression of the work on this subject. Let us cite, for instance, (Benhabib and Baumol (1989), Boldrin and Woodford (1990), Brock (1993) and Brock and Dechert (1991), Brock and Malliaris (1989), Chiarella (1990), Day (1994), Frank and Stegnos (1988), Lorenz (1989), Medio (1993) and Scheinkman (1990)). Other papers that find nonlinearities in macroeconomic data are Nefcti (1984), Potter (1990) and Terasvirta and Anderson (1992). The Fernández Díaz, Escot and Grau paper (2002), investigates, using recursively the BDS and Kaplan tests, whether the foreign exchange rates behave non-linear, studying also the behaviour of the volatility using the ARFIMA models that are able to capture the long memory of this variable.

The results using the correlation dimension for financial series, stock prices and foreign exchange, are also not conclusive, since although the series are longer, such as high-frequency financial data, such data present microstructure. As argued by Hsieh (1991), tick-by-tick data captures micro-market phenomena, such as bid-ask jumps, limit price order executions and others, than will be distortion non-linearity tests. Applications mentioned above to test the financial data can be found in Fernández Diaz et al. (2000), Blank (1992), Brock et al. (1991), De Grauwe et al. (1993), Decoster et al. (1992), Hinich and Patterson (1985), Hsieh (1991), Peters (1991) and Scheinkman and LeBaron (1989). The Evidence of low dimensional chaos is not convincing, but there is evidence of strong non-linear dependence.

The estimation of the largest Lyapunov exponent is a sufficient condition to detect chaos in dissipative systems. Among recent works Bask, Liu and Widerberg (2007) analize stability of electricity prices using Lyapunov exponents. Linton and Shintani (2003) analyze real output time series from various countries, and Serletis and Shintani (2006) investigate monetary aggregates finding in most of the cases statistically negative Lyapunov exponents. Some exceptions are Peters (1991) and Fernández Diaz et al (2002). But, the presence of dynamic noise makes difficult distinguishes between noisy

[7] We can find a general view in: Faggini (2009, pp. 327-340), and Güegan (2009, pp. 89-93).

high-dimensional chaos and randomness. In this regard, a recent article of Hommes and Manzan (2006), using simulations show that a small amount of dynamic noise can result in negative Lyapunov exponents in a chaotic system, even if it is low-dimensional.

Beyond the existence of chaos in financial data, research on them has yielded interesting findings. First, it should be noted as a result of the studies cited above it has been found some structure in the volatility of financial series. In general, the volatility has long memory persistence (Grau-Carles, 2000 and Grau-Carles 2001). The long-range dependence, also known as long memory, is characterized by hyperbolically decaying auto-covariance function, by a spectral density that tends to infinity as the frequencies tend to zero and by the self-similarity of aggregated summands. In many cases, the persistence in volatility is usually detected by a power law.

Power laws

The power laws are regularities that can be found in many financial series, and not just volatility. Power laws are relevant in economics and finance because they are empirical regularities in time series. A good survey on the power laws in economics and finance is the Gabaix (2009) one; some of the power laws are empirically tested the quantity theory of money, company size, city sizes, income and wealth distributions, CEO compensation, stock market activity. Here we will highlight two of the most interesting: The urban dynamics and the stock market activity.

Urban Dynamics is a very important chapter of the applications of chaos and complexity to economic science, considering that change in development is often assumed to be proportional to existing size or scale as

$$\frac{dP_t}{dt} = \lambda P_t \tag{18}$$

where λ denotes a growth rate, expressing the continuous exponential growth by the relation

$$P_t = P_0 \cdot \exp(\lambda\, t) \tag{19}$$

This relation has been used in traditional models of cities to simulate the decline in densities, rents, trips to work, and so on, and also to model population growth in rapidly growing cities. These models are sensitive to their initial conditions in that their ultimate states are likely to be conditioned by their starting values. This sensitivity is characterized by than Arthur (1994), among others, calls *path dependence,* defined as different trajectories that emerge from the application of particular initial conditions. In fact, path-dependent behaviour as well as bifurcations is usually associated with more complex models, where the state is a function of many variables that refer to interacting places and activities (Batty, 2007, pp. 24-29). The types of change affect the entire system, and that in physics is called a phase transition is also characteristic of urban systems qualitative and can be seen at many levels.

Recent works about urban dynamics outline ways in which cities grow to become more complex, moving from simple non-fractal forms to the point where their locational and relational structures of their activities and interactions are poised at equilibrium between order and chaos. With others words, this is the contemporary view of how fractals form, and it relates the morphology of cities to networks, phase transitions, self-organized criticality, and the “edge of chaos” (Batty, 2007, pp. 457-460). In certain measure, at present, urban structure and dynamics may be considered as “fractal geometry”, which allows a more modern and rigorous analysis of emerging ideas in the economics of cities. In effect, as says Batty (2007), in the process of generating theories for cities that are based on their essential dynamics, a new view of systems in general has arisen, and a new paradigm grappling with essence of such systems began to emerge supported by complexity theory and the physics of far-from equilibrium structures. Processes that lead to surprising events, to emergent structures not directly obvious from the elements of their process but hidden within their mechanism, new forms of geometry associated with fractal patterns, and chaotic dynamics, all them are combining to provide theories that are applicable to highly complex systems such as cities (Batty, 2007, p. 5) It is necessary to add that most of the developed models in urban dynamics are cell-based methods that can be categorized as models that use cellular automata, a technique widely applied in simulation of all kinds of spatial issues. In short, the future development of the economics of cities or urban dynamics find a valuable help in the principles and techniques of complexity and chaos, using new kinds of computational models and power laws, whose meaning, applications and interpretation constitutes a very exciting task.

Using large data sets Gopikrishnan et al (1999) and Mantegna and Stanley (1995) established a strong case for inverse “cubic” power law of stock market returns. They established that for several stock market indexes and U. S stocks the law is

$$P(|r_t| > x) \cong x^{-\alpha} \tag{20}$$

where α is in the vicinity of 3 or 4. This implies a kind of universal pre-asymptotic behaviour of financial data at certain frequencies. In the case of volatilities, measuring them as the absolute returns the power law is characterized by

$$Cov(|r_t r_{t-\Delta t}|) \approx \Delta t^{-\gamma} \tag{21}$$

where $\gamma \cong 0.3$ is a rather typical finding, which implies a strong correlation of volatility over time. This dependence can be used for portfolio and risk management. The measurement of long-range dependence is based on estimation of the Hurst exponent or the Detrended Fluctuation Analysis.

Also trading volume and number of traders exhibit power laws, with exponents similar in different types of markets and different countries. Gabaix et al (2003) propose a model to explain these empirical power laws. Their model is based on the hypothesis that large movements in stock market activity arise from the trades of large participants, such as large fund managers, and power laws appears when trading behaviour is performed in an optimal way.

Chaotic modelling in Economics

Besides the literature on the detection of chaos in time series, there are also numerous studies devoted to the modelling of economic phenomena that under certain parameter values can lead to chaotic behaviour. Most work in the field of modelling is based on dynamic models. We refer here to three of which have led to more promising results:

the optimal growth theory, the theory of imperfect competition and modelling of financial markets based on the behaviour of agents.

Let us start with the theory of optimal growth. Growth theory main objective is to explain the fluctuations of the variables around a rising trend. The two dominant paradigms in the theory of growth are models of overlapping generations and the model with infinitely lived person, and both approaches have resulted in models that can generate chaotic behaviour. Grandmond (1985) uses the overlapping generation model to find chaos and cycles.

Most of the work of infinite lived household deal with models that can be reduced to one dimensional difference equation (Benhabib and Nishimura, 1979, Boldrin and Deneckere,1990 and Nishimura and Yano, 1994) that find chaos with different utility functions. Two sector models are studied in Deneckere and Pelikan (1986). Also have been analyzed models with heterogeneous agents (Bewley, 1986) with an equilibrium path that exhibit period two cycles but the endowment is constant over time. Chaos can also occur in oligopolistic models. Kopel (1997), Puu (2000) and Rand (1978) state that the adjustment process of the Cournot model can be chaotic if the reaction functions are non-monotonic and Puu (1991) finds that this happens with isoelastic demand and different marginal costs for the competitor, so it is possible to obtain unimodal reaction functions. A good reference for the review of the investigation of complex oligopoly dynamics is Rosser (2002).

Dynamical systems generated under certain parameter values provoke unstable fluctuations; however, the investigation of chaos control has been applied to Cournot models to try to control this phenomenon. The OGY in Ahmed and Hassan (2000) and Aziga (1999) and the pole placement method in Pyragas (1992) have been used for oligopolies. But those methods present an important drawback when are applied to the oligopoly, and it is that require accurate system information before being implemented. A method that can be used without this large amount of information is the Delayed Feedback Control as in Chen and Chen (2007) that developed an adaptive approach to adjust the marginal costs, establishing how that method can be use in Cournot models to stabilize the market and how it can also benefits smaller companies.

On the other hand, financial market modelling has been able to produce either stable solutions or complex solutions including chaos. Some of these models combine both fundamental variables included in the structural model and non-fundamental factors. These kinds of models assume that agents are one of two types: fundamentalist or chartist. Fundamentalist form their expectations about prices using fundamental factors such as macroeconomic variables. Chartists, however, use information about price history to form their expectations. The interaction between these two types of agents is able to generate price patterns that deviate from fundamentals, although in the long term, fundamental expectations drive prices back to the long-run equilibrium. Research that include this kind of modelling are Day and Huang (1990), Kirman (1991), De Grauwe and Dewachter (1993), Brock and Hommes (1998), De Grauwe and Grimaldi (2005), Fernández Diaz (2000), Lux and Marchesi (2000), Farmer and Joshi (2002), Hommes (2006), and Manzan and Westerhoff (2007).

Economic networks

Inside of this wide field in the set of applications of chaos and complexity, one of the most important approach is that which deal with neural networks, used in economic analysis especially in three different areas. Specifically, the first and most important for predicting time series, the second for the classification of economic agents in categories and the third to model bounded rational economic agents. As Neural Networks are flexible functional forms to approximate any continuous function, they are capable of generating non-linear models to better predict. One of the first contributions in this field is that of White (1988), and other relevant contributions that use more complex networks are Bosarge (1993), Wong (1990), Hiemstra (1996) and Haefke and Helmenstein (1996), who find nonlinearities in time series and improve the predictions' quality. Other contributions with predictions for other macroeconomic variables are the Franses and Draisma (1997) and Swanson and White (1997).

Another ability of neural networks is to classify data will not be linearly separable. The work on which this aspect of neural networks is mainly used is devoted to bankruptcy prediction of economic agents, mainly banks, outstanding in this field contributions from Odom and Sharda (1990), Rahimian et al. (1993) and Tam and Kiang (1992). Other applications with similar results are those of Poddig (1995), Salchenberge et al.(1992) and Altman et al. (1994).

Finally, other interesting application is the use of neural networks to model bounded rational economic agents. In this context, one of the first contributions is that of Sargent (1993). Cho (1994) uses neural networks to model strategies for repeated games; Luna (1996) model the emergence of financial institutions and Orsini (1996) proposed a Neural Network to model the consumption behaviour of individuals.

Now, in a more general context, and always in relation to economic networks, it is necessary to remember that the economy is a complex system in which a large number of interacting agents whose outcome is only observed at the aggregate level and can have unpredictable results as shown in the current crisis. The study of economic networks requires knowledge of a large number of analysis tools such as complexity theory, analysis of time series, the graph theory, game theory and simulation among others.

Economic networks have been studied mainly from two perspectives, one which comes from sociology and economics and the other that comes from the study of complex systems in physics and computer science, always taking as origin. However, in both the network is built in the same way, using nodes that are the agents, companies, banks and countries, and links between the nodes that represent their mutual interaction, debit-credit relations, trade, etc. The complex systems perspective tries to reproduce the observed statistical regularities in the empirical literature, (Albert and Barabasi, 2002). This vision does not seek to find the endogenous behaviour of the agents, but instead they focuses on aspects such as the degree of connectivity or centrality. The predictions obtained from this type of network analysis are provided in aggregate. Structural properties generated by these networks various with stochastic algorithms have been compared with real complex networks. For example, Boss et al (2004), in the context of banking relationships shows that the degree of bank distribution scales as a power law. Similar regularities have been found in international trade. Fagiolo et al (2008) and Garlaschelli and Loffredo (2005) show that the total value of trading in a country scale as a log-normal density. Another noteworthy study is that foreign direct investment also follows a power law (Battiston et al, 2007). In this sense, the main contribution of these studies is the discovery of universal scaling properties as the laws of power.

In contrast, the socioeconomic perspective emphasizes how strategic behaviour of interacting agents is influenced by very simple network architecture. The network is built as a game between competing and collaborating agents Galeotti (2006). The links are added or deleted as a result of decisions that seek to maximize the payoffs. In this context, the agents try to anticipate the behaviour of others with a time horizon, learning from past experiences. This approach relies on game theory and the search of equilibrium results. As pointed out by König et al (2009), Bala and Goyal (2000) and Marsili et al (2004), collaboration and innovation are particularly relevant in this type of analysis. More recently, Farmer and Geanakoplos (2009) and Marsili et al (2004) discussed the formation of networks when the environment is volatility persistent, such as innovation, environmental change and political instability. In this case, the agent rules change while their experiences, so you cannot find an optimal configuration and performance of the system can be very sensitive to changes. But much remains to be done in the study of economic networks. There is no doubt that the combination of individual strategies analysis from the point of view, with the approach of networks and interactions and feedback can give many interesting results. These methods do not always lead to a unique equilibrium but in many cases lead to multiple equilibrium, sudden bifurcations and out-of equilibrium transients that better reflect the real life systems.

The fractal road to economic crisis

To finish, we can characterize the present reality, in the framework of our research, as a fractal road to economic crisis in the following sense. Strange attractors show in relation to statistical predictions and prognostics that randomness is not only provoked by exogenous causes, but also by the very dynamic nature of the system. Both endogenous and exogenous factors can shift a financial system away from equilibrium. As the market reacts, it movies away from stable conditions associated with order, and disequilibria rather than equilibrium are characteristic properties.

Given that the stock market prices $y_1, y_2, \ldots\ldots\ldots y_t$ are not entailed to the elements included in the expectations of the agents, the variable y_t become chaotic, so that the point of equilibrium evolves in the area of limited instability within the strange attractor. Based on the graphic presentations, we arrive to what is named “financial fractal”, a road to explain the present economic crisis. It is necessary to pay attention to the fact that the conventional view in econometrics rest basically in the Gaussian model where the addition of small causes (Central Limit Theorem) constitutes white noise perturbations. However, in the framework of complexity, based in the concept of emergence, the whole is greater than the sum of its parts, and the macroscopic correlations can give rise to feed-back effects with endogenous randomness (chaotic noise) that separates the system of the equilibrium path.[8]

In order to speak about the fractal road to economic crisis is advisable to mention the theory of Black Swan, and related to it, the ideas of Gray Swan and White Swan. The theory of the Black Swan, in our context, affirms that the history is not determined by the predictable, but by what is highly improbable, by events not foreseen that have a

[8] See Nieto de Alba (2009, pp. 212-221).

major impact and that Taleb (2010) named Black Swans. There are some people in the financial world claiming that the colossal global crisis that has occurred in financial markets is an example of a Black Swan event. In other words, it was unpredictable. But as says Melamed (2009, pp. 1-5), if we examine the causes and effects of the actions which produced the current global crisis, we can suggest that they were all pretty well predictable, and because of this we must think that they were more like Gray Swans, and avoidable. Let us remember, finally, that White Swan means order, regularity and stability, in contrast with the disorder, irregularity and instability of the Gray Swan in the global evolution of the system.

The point of departure of the financial crisis may be found in the collective stock exchange hysteria on Monday 21 January of 2008, due to the fear of one recession in the United States economy after the beginning of subprime mortgage crisis. The lack of payment of these mortgage provoked by the high rates of interest (more that 5% in 2007) gave rise to an impact in cascade over the set of banking balances, touching fully to the first bank of credit and also to all the entities in which the investors had gotten securitized credits. Crisis was extended to the rest of the world with heavy problems of liquidity among banks what, in his stead, has produced a strong credit crunch, so that a sensible loss of confidence in the banking system.

Following with our metaphor, this means that we are in the presence of a Black Swan that is situated in the middle of the global lake. But it is necessary to point out that the present crisis is the consequence of applying a conventional monetary and financial policy, with a lax control and a clear lack of regulation within the financial system, together with a high level of speculation and corruption. The *London School of Economics Report* issued in 2010 states that the financial sector has undergone an astonishing roller-coaster ride in the course of a decade, ascending to heaven and subsequently descending to hell, as in the 1930s. To sum up, we are in front of a Black Swan, what makes us remember the famous Tosca´s sentence in the presence of lying Baron Scarpia: "E avanti a lui tremava tutta Roma". Instead of trembling, it was necessary to react immediately undertaking a set of deep structural reforms, mainly in the financial system and its international coordination. But at the end, when a decision was make on how to manage the crisis, we came to have again *more of the same*, that is, the conventional economic wisdom, suitable only for the case of the White Swans.

Returning to a world of nonlinearities, complexity, in dependence of the initial conditions, with irregularities and far from equilibrium, that is, not to an ideal world but to the real one, there is no other solution that to find the Gray Swan and to profit by possibilities derived of the "fractal geometry" developed by Benoit Mandelbrot[9]. If it is true that fractality don't convert the Black Swan in a predictable happening, is enough to work and take decisions, and become it (the Black Swan) in a Gray Swan. In a non-linear equation, the growth of the parameter allows us entering in chaotic behaviour, to come off it and to turn to chaos at a higher level of complication, something characteristic of the non-linear dynamic systems. Now order appear as intermittences of a strange attractor that capture the new realities and values of environment in the global evolution of the system. All that provides a scientific model for a preventive and complex management in what the instability and the disequilibria constitute self-organized process leading to a new order. This process has not happening in the crisis that at present devastates our economies.

[9] See Fernández Díaz (2000, pp. 121-124).

Controlling chaos

Before to conclude it is necessary a short reference to the control of chaos, a subject that, in the last years, call the attention of the researches about chaos. The idea of controlling chaos has been first formulated by Ott, Gregobi, and Yorke in 1990, and attracted great interest among physicists over the past one and half decade. There are situations in which the chaotic behaviour is not desiderated. In these cases is where the control of chaos results outstanding and advisable. Very often the control of chaos is defined as that set of techniques that have as objective to cancel and/or to stabilize the chaotic behaviour. In fact, the chaos control search to stabilize any of the multiple periodic or quasi-periodic equilibrium existing in the dynamic that became unstable but is embedded in the stranger and chaotic attractor (Gandolfo 2009). With other words, we can say that controlling chaos, known sometime as "survival method", consist in actuate in a chaotic dynamic process with the purpose of reaching a determined objective relative to the behaviour of the system. For instance, the control should be able to work for avoiding a collapse in an ecologic process, or to correct periodic orbits in chaotic systems.

There are different techniques of controlling chaos: the OGY (Ott, Gregobi, Yorke), that consider the perturbation of the parameters of control for stabilizing unstable periodic orbits existent into the attractor; the Targeting Method (Shinbrot et al 1990) consistent in the global control of chaos that, in his turn, is divided in others techniques, as the Forward Targeting Method and the Forward-Backward Method; the GM Method (Güemez and Matías 1993), that search to stabilize orbits of arbitrary period *p* in a chaotic system applying periodic perturbations with the same period, being the principal advantage of this method the fact that is not necessary to know *a priori* the dynamic of the system, neither analytic nor numerically. Besides, we have the DFC Method (Delayed Feedback Control), (Pyragas 1992), applied to systems in continuous time, that consist in aggregating a linear sign of feedback to the variable of the system, and the CF (Constant Feedback Method), (Parthansarathy 1995 and Wieland 2002), applied to systems in discrete time). We do not enter in this article in more details about the different ways or techniques for controlling chaos[10], but it is interesting to point out the possibility of applying them in Economics[11], and more concretely in the field of Economic Policy. Effectually, the policy-maker takes decisions using a dynamic model with "control variables" and "variables of state", and if the dynamic model is non-linear and chaotic, then we can handle the variables of control to modify the state variables and, consequently, the objectives and evolution of the economic system. In this framework, the principal implication of these techniques is that the economic policy of "fine tuning" turn up to have sense, unlike of the traditional and orthodox theory that had fully rejected this type of instruments of measures based on the hypothesis of the

[10] See for the study of these techniques: Sanjuán and Gregobi (2010).

[11] There are many contributions that have applied control of chaos in non-linear dynamic economic models. Holyst (1996); Holyst, Hagel, Haag and Weidlich (1996); Holyst, Hagel, and Haag (1997); Holyst and Urbanowicz (2000); Ahmed and Hassan (2000); Salarieh and Alasty (2009), and Chen and Chen (2007) applies the control of chaos to microeconomic models of business sector competition. Haag, Hagel and Sigg (1997) use techniques of control to stabilize urban economic system. Kaas (1998); Bala, Majumdar and Mitra (1998), and Kopel (1997) has applied control methods to disequilibrium dynamics models. Mendes and Mendes (2008) apply the OGY method in a model of endogenous cycle of overlapping generations. Finally, Wieland and Westerhoff (2005) shows as may utilize the OGY, DFC and CF to stabilize the evolution of the rates of exchange.

rational expectations, hypothesis that, of fact, by definition, become inconsistent when models of chaotic behaviour are used.

CONCLUDING REMARKS

Of all that we have expounded on, within the framework of modern theories of chaos and complexity, some conclusions of interest can be extracted in relation to new approaches and techniques in the field of Economic Science. First of all, it is convenient to highlight, as can be easily deducted, that the importance of the application of the theory of chaos in our scientific field resides in the fact that it extends beyond the frontiers delineated for probabilistic analysis, by attempting to offer an consistent explanation of endogenous nature, to that part of behaviour which is habitually excluded on adopting a resigned attitude, refusing to interpret it as an inexplicable error, or a shock, or an unpredictable impulse. Besides is also fundamental and imperative to persevere with the intrinsically dynamic character of economic phenomena, which involves assuming the need of a feed-back or retroactive mechanism to facilitate the making of decisions, with a greater degree of rationality. Too it appears to be clear that economic phenomena are habitually non-linear in nature, show a great sensibility to modification of the initial conditions, and often exhibit irregular behaviour.

A chaotic sequence and a random sequence look superficially the same, even though they are very different. Unlike the random sequence, the chaotic one is completely deterministic, and contains a good deal of hidden order. The use of methods to discern that order, distinguishing between chaos and randomness, can, for instance, deepen the understanding of the dynamics of financial markets and may lead to an improvement in short-term price predictability. Let us remember, among others, the long-memory, the correlation dimension, the embedding dimension, the Lyapunov exponent, the Kolmogorov entropy, and the BDS and Kaplan tests.

It is important to bring to light that, given the complexity of the economic universe, the great difficulty of any type of prediction of the future is evident, but this does not decrease our possibilities of intervention. It could even be said, that these possibilities increase notably from the moment that, on not being trapped by an established mechanism from the start, we are free to explore alternative future trajectories. On the other hand, let us remember that emergence bear a great relation with complexity and chaos, and it can help to find underlying laws that we are searching behind the irregularity.

Complexity, as science of complex systems, implies a challenge to integrate a large number of disciplines, as non-linear dynamics and theory of chaos, the statistical physics, stochastic process theory, the theory of information, the networks theory, the biology and the computational sciences. As highlighted in the Review Nature in the year 2005, there is a new line of research, named “synthetic biology” that integrates different scientific areas, just as the non-linear dynamic, the physic of complex systems, the engineering and the molecular biology. This emergent field of research is intrinsically interdisciplinary and constitutes an advance to take into account for the next years in our work about the application of chaos and complexity theories. The studies of complex networks in physics, mathematics, economics and other sciences, is also at present undergoing a very important development, which represents new possibilities in the task that we are tackling. It is clear that in the applications of chaos and complexity theories to Economics we use the methods of other disciplines (principally those of the mathematical, physical and biological sciences) wherever they seem to fit, although one must tailor their usage to one´s own discipline.

Situated concretely in the very sphere of Economics, besides the different concepts and techniques necessaries to understand the role, significance and functioning of complexity and chaos, we have seen the main and most recent applications in our field. Now, as summary or conclusions, we are going to emphasize some of them, pointing out its degree of efficacy, real interest and capacity of explanation.

Among the applications to Economics we must outstands those related to capital market, because the time series are longer and is easier, most profitable and useful to work with them. At present, the Efficient Market Hypothesis is not well supported by empirical evidence (Fernández Díaz and Grau, 2011), and has often failed to explain the market behaviour. At the same time, markets are not well described by random walk model. In the last part of this article we have done reference to the Madrid Stock Exchange case to detect chaotic behaviour in time series from January 1941 to January of 1998, for the General Index, with 684 monthly data, and from January of 1987 to March of 1998 to the daily IBEX 35 Index, handling a set of 2.776 data. Some paper (above-mentioned) finds that nonlinearities in financial assets can be product of contamination produced by shifts in the distribution of the data. Using the BDS and Kaplan tests it is shown that, some of the nonlinearities found in foreign exchange rate returns, can be the product of shifts in variance while other do not. Also, the behaviour of the volatility is studied, showing that long-range correlation modelling is able to capture long memory, but depending on the *proxy* used for the volatility, is not always able to capture all the nonlinearities of the data.

We have seen previously applications of the techniques of chaos in modern books about urban dynamic that may be considered as “fractal geometry”, a new way of getting a more deep and productive analysis of the structure and evolution of cities from an economic point of view. It is important to emphasize that we find in urban dynamics and in the stock market, between other fields, the existence of power laws. Other type or application is the very well known ZIPF´s Law. In short, it says that for most countries the size distribution of cities strikingly fits a power law: the number of cities with population greater than S is proportional to 1/S. The importance of this law is that, given the very strong empirical support, it constitutes a minimum criterion of admissibility for any model of local growth, or any model of cities, and in self-organization and hierarchic structure process in the space. It is interesting to point out that power laws are related to changes of phases or to critical states, where a many body system evolves from one state of self-organization into a different one. Often these states are complex and they indicate patterns of self-organization in the evolution of complexity. Power laws show-up as indicators of ways into which a given system responds to the environment and evolves its organization. The existence of a power law can be thought of as due to a simple physical principle: scale invariance. Because the growth process is the same at all scales, the final distribution process should be scale-invariant, what forces it to follow a power law. As Gabaix (2009) remembers, despite or perhaps because their simplicity, scaling questions continue to be fecund in generating empirical regularities that, sometimes are among the most surprising in the social sciences. They in turn motivate theories for their explanation, which often require new ways to view economic issues.

In the framework of our research, it is necessary to stand out the possibilities of applying the Artificial Neural Networks (ANN) to business, Economics and Finance. For instance, there are multivariable model from genetic algorithms and artificial neural networks to predict the sign of the variations of stock-market indexes. Artificial Neural Networks are composed of a large number of highly interconnected processing elements working in unison to solve specific and complex problems. In more practical terms Neural Networks are non-linear statistical data modelling or decision making tools. If we speak about the economic application, we find that Neural Networks can be used as an alternative to more traditional methods such as discriminant analysis or logistic regression. A special feature of Neural Networks that distinguishes them from traditional methods is their ability to classify data which are not linearly separable. The majority of papers that use Neural Networks for classification tasks in Economics can be found in the area of bankruptcy of economic agents, mainly banks. But probably, we find the largest share of economic applications of Neural Networks in the field of prediction of time series in the capital markets. As the efficient market hypothesis is neither correct nor generally accepted, is advisable to use non-linear models to improve the fit and thus the prediction, taking into account that Neural Networks are flexible functional forms that allow providing effective non-linear models. To end up, it is convenient to remember that, in the comparisons of available tests for nonlinearity and chaos we have the White´s proposed Neural Network test.

Recent works deal with the applications of the concepts of fractal and strange attractor to explain the present economic crisis, considering that in a world of nonlinearities, complexity, depending of the initial conditions, with irregularities and far from equilibrium, there is no other solution that to profit the possibilities derived of the fractal geometry. Although, as we know, this don´t convert the Black Swan in a predictable happening, is enough to work and take decisions, and become it in a Gray Swan, that is, in a new approach that avoids more of the same, and allows a better management of the crisis, leaving the conventional economic wisdom, and undertaking a set of deep structural reforms, mainly in the financial system and its international coordination.

Finally, we must not forget the great importance that has achieved in the last years, as we have seen, the specific chapter of controlling chaos applied to the field of Economic Science, finding a clear parallelism between the control of perturbations to cancel or to stabilize the chaotic behaviour, and the problem in economic policy of getting the necessary instruments or “control variables” for doing possible the “fine tuning” approach to reach the objectives established by the policy-maker. The development of this point within the applications of Theory of Chaos to Economics constitutes certainly one of the most fructiferous research tasks at present.

To conclude about the possibilities and usefulness of applications of chaos theory to Economics at present and in the future, it is necessary to distinguish between the pessimistic and apathetic approaches, that only pay attention to identify chaos with unpredictability, and those that tries to deep and find the way through which nonlinear dynamic and chaos can help the economy as an evolving complex system. As we say, the first group, that is, the frigid one, principally the econometricians, argues that is not possible the predictability, and because of it chaos theory don´t works, but the fact that we cannot make exact predictions of the long-term behaviour of chaotic systems does not exclude the possibility of making, more or less accurate, short-run forecasts.

In the presence of this duality, darkness or light, pessimism or optimism, allow us the license of finishing our work with a relevant reference to the best literature, very illustrative and opportune, remembering that , between the unexplored limbo of fear and sadness perceived in the Henry James´s verses of his work “The turn of the Screw”, and the best of the worlds of Candide, the Voltaire´s famous story, we find ourselves with a present and future reality that belong to us and that is of our absolute and exclusive responsibility. Let us catch the message and meet the challenge, applying it to our subject.

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Universidad Complutense de Madrid
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