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Values of games with weighted graphs ^{*}

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Abstract

In this paper we deal with TU games in which cooperation is restricted by means of a weighted network. We admit several interpretations for the weights of a link: capacity of the communication channel, flow across it, intimacy or intensity in the relation, distance between both incident nodes/players, cost of building or maintaining the communication link or even probability of the relation (as in Calvo et al., 1999). Then, accordingly with the different interpretations, we introduce several point solutions for these restricted games in a parallel way to the familiar environ of Myerson. Finally, we characterize these values in terms of the (adapted) component efficiency, fairness and balanced contributions properties and we analyze the extend to which they satisfy the stability property.

Key words: TU-game, weighted graph, Myerson value, fairness, balanced contributions, stability.

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1 Introduction

A cooperative TU game describes a situation in which several actors can obtain certain transferable payoffs by means of the cooperation. Mathematically a TU

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game is a set of players and a characteristic function that assigns to each subset of players (coalition) a real number representing the worth of the coalition.

In this paper we deal with TU games in which the cooperation is restricted by means of a weighted network. The seminal work on games in which restrictions in the cooperation are given by a graph is debt to Myerson (1977). He assumed that the nodes in the graph are the players in the game, each link representing a direct bilateral communication channel. From this starting point, he defined the graph-restricted game and proposed as a point solution for players in this environ the so called Myerson value, i.e., the Shapley value of the restricted game. Moreover he characterized this value in terms of components efficiency and fairness. Later, Myerson (1980) gave another characterization of his value substituting fairness by balanced contributions.

In the Myerson model, bilateral restrictions in the communication are dichotomous: they exist or they do not. Nevertheless, as has long been appreciated, each direct connection can be only partially limited. This leads to use a weighted graph as a model of the restrictions in the communications. A weighted graph consists of a set of nodes and a set of links, each link having an associated weight, a non-negative real number that can be interpreted in different ways: the capacity or the capability of the communication channel, the flow across it, the degree of intimacy, intensity or frequency if the link represents a social relation, the distance between both incident nodes, or even the cost of building or maintaining the communication link. Calvo et al. (1999) introduced a probabilistic model in which the weight of a link is the probability of to establish the relation, these probabilities being independent. Later, Gómez et al. (2008) generalized this probabilistic model dropping out the independence hypothesis. In a related context, Jiménez-Losada et al. (2010) consider cooperative TU games with Choquet players in which the restrictions in the communications are modeled using an (undirected) fuzzy graph¹.

In this paper we introduce point solutions for games with restrictions in the communications modeled by a weighted graph, following a parallel way to the familiar territory of Myerson.

In a TU game, each set of players is able to obtain its total dividend (Harsanyi, 1959). The Myerson underlying idea is that, under restrictions in the communications every coalition that can be connected in the graph (eventually using intermediaries) obtains all its dividend in the game. Otherwise, i.e., in absence of connectedness, its dividend vanishes. And thus, the binary framework of communication is projected in an all or nothing possibility of obtaining the dividend. Nevertheless, the weighted graph introduces a non dichotomous but fuzzy scheme of relations. In accordance with this, we propose to consider a

¹ For details on fuzzy graphs, reader can see Mordeson and Nair (2000)

weighted-graph game in which players weighted-link connected obtain a part but not all their dividend and the lack of connectedness leading, of course, to loose the dividend. We introduce several forms of calculating this fraction of the dividend taking into account various alternative interpretations of the weights. Then, we use the Shapley value of the corresponding weighted-graph restricted game as an appropriated solution concept for this situation and we characterize it in terms of the (adapted to this framework) component efficiency, fairness and balanced contributions properties. Finally, we explore the extent to which the defined values satisfy the stability property.

The remaining of the paper is organized as follows. In Section 2 we introduce some notation and preliminaries; in Section 3 we define the weighted-graph restricted games and the different extensions of Myerson value accordingly with the various interpretations of the weights. Section 4 is devoted to characterize the defined values. In Section 5 we deal with the stability property and the paper ends with a section of final remarks and conclusions.

2 Preliminaries

A cooperative n -person game or a TU-game is a pair (N, v) , $N = \{1, \dots, n\}$ being the set of players and v , the characteristic function, a map $v : 2^N \rightarrow \mathbb{R}$ satisfying $v(\emptyset) = 0$. For each coalition $S \subset N$, $v(S)$ represents the transferable utility that S can obtain whenever its members cooperate.

We will note G^N the set of all TU-games with players set N . It is easy to see that G^N is a vector space. The game (N, u_S) , $\emptyset \neq S \subset N$, with characteristic function given by:

$$u_S(T) = \begin{cases} 1, & \text{if } S \subset T \\ 0, & \text{otherwise} \end{cases}$$

is known as the unanimity game corresponding to S . The family of all games $\{(N, u_S)\}_{\emptyset \neq S \subset N}$ is a very useful basis of G^N . As a consequence, the characteristic function v of every game in G^N can be written as a linear combination of unanimity characteristic functions:

$$v = \sum_{\emptyset \neq S \subset N} \Delta_v(S) u_S.$$

The coefficients (coordinates) of v in such a basis are known as Harsanyi dividends (Harsanyi, 1959). The worth of every coalition S can be written in

terms of its Harsanyi dividends. For each $S \subset N$, $S \neq \emptyset$:

$$v(S) = \sum_{\emptyset \neq T \subset S} \Delta_v(T).$$

A very popular point solution for TU-games is the Shapley value (Shapley, 1953), which assigns to every player the following convex linear combination of his marginal contributions to different coalitions:

$$Sh_i(N, v) = \sum_{S \subset N \setminus \{i\}} \frac{(n-s-1)!s!}{n!} (v(S \cup \{i\}) - v(S)), \quad i \in N.$$

An alternative expression for this value in terms of the dividends is:

$$Sh_i(N, v) = \sum_{i \in S \subset N} \frac{\Delta_v(S)}{s!}.$$

A graph or a network is a pair (N, γ) , $N = \{1, 2, \dots, n\}$ being the set of nodes and γ a subset of $\gamma_N = \{\{i, j\}, i, j \in N, i \neq j\}$. Each link $\{i, j\} \in \gamma$ represents a direct relation or a communication channel between i and j . Γ^N denotes the set of all graphs with nodes set N .

Given a graph $(N, \gamma) \in \Gamma^N$, we will say that two nodes i and j are directly connected in γ if $\{i, j\} \in \gamma$. And we will say they are connected in γ if it exists a sequence of nodes i_1, i_2, \dots, i_k with $i_1 = i$, $i_k = j$ such that $\{i_l, i_{l+1}\} \in \gamma$, for $l = 1, \dots, k-1$. A set $S \subset N$ is connected in γ if every pair of nodes in S is connected. A connected component, C , of the graph (N, γ) is a maximal connected subset. That is, C is connected and, for all $C' \subset N$, if $C \subset C'$ and $C \neq C'$, then, C' is not connected. A graph (N, γ) induces a partition N/γ of the set N in connected components.

Given a set $S \subset N$ and a graph (N, γ) , the restriction of the graph γ to the set S is the graph $(S, \gamma|_S)$. We will note S/γ the set of the connected components of S in $(S, \gamma|_S)$. A subgraph of a graph (N, γ) is a graph (N, γ') with $\gamma' \subseteq \gamma$.

Given a graph (N, γ) and a link $l \in \gamma$, $(N, \gamma \setminus \{l\})$ is the subgraph obtained when the relation l is severed and (N, γ_{-i}) is the resulting subgraph when all the incident in i links are broken and then i becomes an isolated node in the resulting graph.

Given $(N, \gamma) \in \Gamma^N$ and $(R, \gamma|_R)$, its restriction to $R \subseteq N$, we will call connection subgraph of $S \subseteq R$ in $(R, \gamma|_R)$ to each graph $(D(\eta), \eta)$ such that:

i) $\eta \subseteq \gamma|_R$ and S is connected in η

ii) $D(\eta) = \{i \in N \text{ such that it exists } j \in N \text{ with } l = \{i, j\} \in \eta\}$

A connection subgraph of $S \subseteq R$ in $(R, \gamma|_R)$ is minimal if there is no other connection subgraph $(D(\eta'), \eta')$ of S in $(R, \gamma|_R)$ with $\eta' \subsetneq \eta$. Given a graph (N, γ) and $S \subseteq R \subseteq N$ we will denote $\mathcal{CG}(S, R, \gamma)$ (respectively, $\mathcal{MCG}(S, R, \gamma)$) to the family of all (minimal) connection graphs of $S \subseteq R$ in $(R, \gamma|_R)$.

A communication situation is a triple (N, v, γ) , (N, v) being a TU game and (N, γ) a graph. The set of all communication situations with players-nodes set N will be noted \mathcal{CS}^N . An allocation rule ψ on \mathcal{CS}^N is a map $\psi : \mathcal{CS}^N \rightarrow \mathbb{R}^n$, $\psi_i(N, v, \gamma)$ representing the outcome for player i in game (N, v) given the restrictions in the communication imposed by the graph (N, γ) .

The Myerson value (Myerson, 1977) is the allocation rule μ on \mathcal{CS}^N defined by:

$$\mu(N, v, \gamma) = Sh(N, v^\gamma), \text{ where } v^\gamma(S) = \sum_{C \in S/\gamma} v(C), \text{ for all } S \subseteq N.$$

Myerson (1977) characterized this allocation rule in terms of component efficiency (for all $C \in N/\gamma$, $\sum_{i \in C} \mu_i(N, v, \gamma) = v(C)$) and fairness (for each $l = \{i, j\} \in \gamma$, $\psi_i(N, v, \gamma) - \psi_i(N, v, \gamma \setminus \{l\}) = \psi_j(N, v, \gamma) - \psi_j(N, v, \gamma \setminus \{l\})$). He also characterized (Myerson, 1980) this allocation rule in terms of component efficiency and balanced contributions (given $i, j \in N$, $\psi_i(N, v, \gamma) - \psi_i(N, v, \gamma_{-j}) = \psi_j(N, v, \gamma) - \psi_j(N, v, \gamma_{-i})$).

3 Values for weighted communication situations

In this section we admit several interpretations for weights in a graph and we accordingly define several weighted-graphs restricted games for players involved in a TU game with restrictions in the connections given by such a weighted network. Finally, we propose several values that generalize the classical Myerson one.

3.1 Weighted graphs

Definition 3.1 *A weighted graph or a weighted network is a pair (N, γ_w) , $N = \{1, \dots, n\}$ being a set of nodes and $\gamma_w = \{\gamma, \{w_l\}_{l \in \gamma}\}$, with $\gamma \in \Gamma^N$ and the weights $w_l \geq 0$, for all $l \in \gamma$. We will denote Γ_w^N the set of all weighted graphs with nodes set N .*

The weights w_l in a weighted network can admit many different interpretations. If the network is viewed as a model of communications or transport among its nodes, weights can be interpreted as the capacities of the communication channels and then, we would assume $w_l \in (0, 1]$ for all $l \in \gamma$.

Alternatively, weights can be viewed as flow values in the network and so $w_l \in [0, \infty)$ (or $w_l \in [0, k_l]$, if we suppose that each channel has a bounded flow). Sometimes, the network will describe the social relations among actors and in this case, a reasonable meaning of weights is that they measure intensity, intimacy or frequency of bilateral relations. So, for that situations, we will assume $w_l \in (0, 1]$. Weights can also represent distances between nodes or even costs of to create or to maintain links. In this last case, $w_l \in [0, \infty)$, for all $l \in \gamma$. Calvo et al. (1999) propose to interpret each weight as the probability of the corresponding relation and to assume the links independence.

We will say that two nodes $i, j \in N$ are directly connected (connected) in the weighted graph (N, γ_w) if they are directly connected (connected) in the graph (N, γ) . As a consequence, the set of connected components in (N, γ_w) , that we will call $N|\gamma_w$, coincides with $N|\gamma$.

We will note $(N, \gamma_w \setminus \{l\})$ the weighted subgraph of (N, γ_w) given by $(N, (\gamma \setminus \{l\})_{w \setminus \{w_l\}})$ in which all the links $l' \neq l$ have the corresponding weight in (N, γ_w) . We will also note (N, γ_w^{-k}) the weighted subgraph obtained from (N, γ_w) when deleting all the incident in k links and, obviously, its associated weights, i.e., $(N, \gamma_w^{-k}) = (N, (\gamma_{-k})_{\{w_l\}_{l \in \gamma_{-k}}})$.

3.2 Weighted-graph restricted games

Definition 3.2 *A weighted communication situation is a triplet (N, v, γ_w) in which (N, v) a TU-game and (N, γ_w) is a weighted network, the nodes in the network being the players in the game.*

In accordance with the different meanings of weights, we can distinguish four particular families in \mathcal{WCS}^N , the set of all weighted communication situations with players set N : the classes $\mathcal{WCS}^{N,c}$, $\mathcal{WCS}^{N,d}$, $\mathcal{WCS}^{N,f}$ and $\mathcal{WCS}^{N,p}$, respectively including the c -weighted communication situations (c for capacity) in which weights represent capacities of channels or intimacy in the relations, the d -weighted communication situations (d for distance), the f -weighted communication situations (f for flow) and the p -weighted communication situations (p for probability).

Definition 3.3 *Given (N, v, γ_w) a weighted communication situation, we define the weighted-graph restricted game as the TU-game in G^N with characteristic function:*

$$v^{\gamma_w}(S) = \sum_{R \in \mathcal{S}|\gamma} v^{\gamma_w}(R)$$

where for $R \in S|\gamma$,

$$v^{\gamma w}(R) = \sum_{T \subset R} \Delta_v(T) \alpha_T^R(\{w_l\}),$$

$\alpha_T^R(\{w_l\}) \in [0, 1]$ being the proportion of the dividend $\Delta_v(T)$ that the coalition $T \subset R$ retains as a consequence of the communication channels capacities, or because of the difficulties to establish connections created by costs, distances or flows, or because of the intensity or frequency in the relations. When there is no ambiguity with respect the weights $\{w_l\}_{l \in \gamma}$ we will simplify the notation writing α_T^R instead of $\alpha_T^R(\{w_l\})$.

For each interpretation of the weights w_l , $l \in \gamma$ we will try to define this α_T^R , for $T \subset R$ in a consistent way. Let $\{\eta_1^{T,R}, \eta_2^{T,R}, \dots, \eta_{t(R)}^{T,R}\} = \mathcal{MCG}(T, R, \gamma)$ ² the family of all minimal connection graphs of $T \subseteq R$ in $(R, \gamma|_R)$. Then we propose:

- If $(N, v, \gamma_w) \in \mathcal{WCS}^{N,c}$, w_l represents the capacity or the intensity or the intimacy of link l . Then:

$$\alpha_T^R = \max_{i=1, \dots, t(R)} \min_{l \in \eta_i^{T,R}} \{w_l\}, \text{ for } |T| \geq 1, \text{ and } \alpha_T^R = 1, \text{ otherwise.}$$

In the previous expression it is assumed that the possibilities of communication or transport are bounded by the capacity of the channel (the minimal capacity of its links). If several alternatives are available for players, then they will prefer the one in which the minimal capacity is maximal. Similarly it seems natural to assume that the intimacy or confidence among several actors is at most the minimal of the bilateral ones. When a set of actors want to relate themselves they choose among the possible intermediaries those ones such that the minimal total intimacy (or confidence) is maximal.

- If $(N, v, \gamma_w) \in \mathcal{WCS}^{N,d}$, w_l represents the distance between the two incident nodes in the link l or the cost of to create or to maintain this link. Then:

$$\alpha_T^R = \max_{i=1, \dots, t(R)} \left\{ \frac{1}{1 + \sum_{l \in \eta_i^{T,R}} w_l} \right\}, \text{ for } |T| \geq 1, \text{ and } \alpha_T^R = 1, \text{ otherwise}^3.$$

² The same definitions will be obtained if, instead of $\mathcal{MCG}(T, R, \gamma)$, we use $\mathcal{CG}(T, R, \gamma)$ or even any subset of $\mathcal{CG}(T, R, \gamma)$ containing $\mathcal{MCG}(T, R, \gamma)$. For details, see Gómez et al. (2004).

³ Besides α_T^R depends on the meaning of the weights, this dependence will be notationally ignored.

Here, we assume that the communication possibilities decrease with the distance or the cost. If several minimal connection graphs are available, then the geodesic (minimal total distance) or minimal cost ones are preferred.

- If $(N, v, \gamma_w) \in \mathcal{WCS}^{N,f}$, w_l represents the flow between the two incident nodes in the link l . Then:

$$\alpha_T^R = \max_{i=1, \dots, t(R)} \left\{ \frac{1}{1 + \max_{l \in \eta_i^{T,R}} w_l} \right\}, \text{ for } |T| \geq 1, \text{ and } \alpha_T^R = 1, \text{ otherwise.}$$

Here, we assume that the communication possibilities decrease with the flow (or the traffic intensity). If several alternatives are possible then the one in which the traffic jam is minimal will be preferred.

- If $(N, v, \gamma_w) \in \mathcal{WCS}^{N,p}$, we are in fact considering the model in Calvo et al. (1999), and thus:

$$\alpha_T^R = \sum_{i=1}^{t(R)} \prod_{l \in \eta_i^{T,R}} w_l - \sum_{i < j} \prod_{l \in \eta_i^{T,R} \cup \eta_j^{T,R}} w_l + \dots + (-1)^{t(R)+1} \prod_{l \in \bigcup_{j=1}^{t(R)} \eta_j^{T,R}} w_l,$$

for $|T| \geq 1$, and $\alpha_T^R = 1$, otherwise.

In this case, α_T^R can be interpreted as the probability of connecting T with links between members of R in a random selected graph with independent links which probabilities are the weights.

Remark 3.1 The Myerson approach can be considered as a particular case of each one of the previous situations. In fact, if for $(N, v, \gamma_w) \in \mathcal{WCS}^{N,c}$, $w_l = 1$ for all $l \in \gamma$, so that the capacities are not restricted, then we find the un-weighted graph case of Myerson. Analogously, for $(N, v, \gamma_w) \in \mathcal{WCS}^{N,d}$, the case of null distances or costs leads to the Myerson model. Similarly, for $(N, v, \gamma_w) \in \mathcal{WCS}^{N,f}$ the case of null flow or traffic intensity gives the Myerson model. Finally, it is known that probabilistic communication situations generalize the deterministic case of Myerson in which links are dichotomous. In all of these cases, given a weighted communication situation (N, v, γ_w) , for all $T \subseteq R \subseteq N$ with R connected in γ , $\alpha_T^R = 1$ holds. As a consequence, \mathcal{CS}^N , the set of all determinist communication situations can be viewed as a subset of $\mathcal{WCS}^{N,c}$, $\mathcal{WCS}^{N,d}$, $\mathcal{WCS}^{N,f}$ and $\mathcal{WCS}^{N,p}$.

Remark 3.2 The given definitions of α_T^R are consistent also with the idea of continuity in the following sense. Given (N, v, γ_w) and $(N, v, (\gamma_w \setminus \{l^*\})) \in \mathcal{WCS}^{N,c}$ (or $\in \mathcal{WCS}^{N,p}$), then for all $T \subseteq R \subseteq N$, $\alpha_T^R(\{w_l\}) \rightarrow \alpha_T^R(w \setminus \{w_{l^*}\})$ if $w_{l^*} \rightarrow 0$. Similarly given (N, v, γ_w) and $(N, v, (\gamma_w \setminus \{l^*\})) \in \mathcal{WCS}^{N,d}$ (or $\in \mathcal{WCS}^{N,f}$) then, for all $T \subseteq R \subseteq N$, $\alpha_T^R(\{w_l\}) \rightarrow \alpha_T^R(w \setminus \{w_{l^*}\})$ if $w_{l^*} \rightarrow \infty$.

In the next propositions, given a weighted graph, we obtain the corresponding restricted game for each unanimity one.

Proposition 3.1 *Let $(N, u_T, \gamma_w) \in \mathcal{WCS}^{N,c}$, $T \subseteq N$. If $\mathcal{MCG}(T, N, \gamma) = \{\eta_1^T, \dots, \eta_{t(N)}^T\}$ ⁴ and $\beta_{\eta_i^T} = \min_{l \in \eta_i^T} \{w_l\}$ for $i = 1, \dots, t(N)$, then:*

$$i) \ u_T^{\hat{w}} = \sum_{i=1}^{t(N)} \beta_{\eta_i^T} u_{D(\eta_i^T)} - \sum_{i < j} \min\{\beta_{\eta_i^T}, \beta_{\eta_j^T}\} u_{D(\eta_i^T \cup \eta_j^T)} + \dots +$$

$$+ (-1)^{t(N)-1} \min_{i=1, \dots, t(N)} \{\beta_{\eta_i^T}\} u_{D(\cup_{i=1}^{t(N)} \eta_i^T)}, \text{ if } |T| > 1 \text{ and } \mathcal{MCG}(T, N, \gamma) \neq \emptyset$$
(1)

$$ii) \ u_T^{\hat{w}} = 0, \text{ if } |T| > 1 \text{ and } \mathcal{MCG}(T, N, \gamma) = \emptyset$$

$$iii) \ u_T^{\hat{w}} = u_T \text{ if } |T| = 1.$$

Proof: To prove *i*), consider $S \subseteq N$. If $T \not\subseteq S$ then, by the definition of $u_T^{\hat{w}}$, $u_T^{\hat{w}}(S) = 0$ as no coalition with dividend different from zero is contained in S . On the other hand, the set of nodes of each minimal connection graph of T in N contains by definition all nodes in T and thus the characteristic function in the right hand term of (1) evaluated in S gives also 0.

Let us then consider the case in which $S \subseteq N$ is such that $T \subseteq S$ and S is connectable in γ , i.e., S is a subset of a connected component of N in γ . If S is not connectable in γ , we would consider the intersection R of S with the connected component of N in γ containing T and we would suppose that this intersection still contains T . Otherwise the previous reasoning applies and both members in (1) applied to R give zero. So, let us suppose that $\mathcal{MCG}(T, S, \gamma) = \{\eta_1^{T,S}, \dots, \eta_{t(S)}^{T,S}\}$. Of course $\mathcal{MCG}(T, S, \gamma) \subseteq \mathcal{MCG}(T, N, \gamma)$. Then,

$$u_T^{\hat{w}}(S) = \alpha_T^S(\{w_l\}) = \max_{i=1, \dots, t(S)} \min_{l \in \eta_i^{T,S}} \{w_l\}$$

and, assuming without loss of generality that $\eta_1^{T,S}, \dots, \eta_{t(S)}^{T,S}$ are ordered so that $\beta_{\eta_1^{T,S}} \leq \beta_{\eta_2^{T,S}} \leq \dots \leq \beta_{\eta_{t(S)}^{T,S}}$,

$$\begin{aligned} & \left(\sum_{i=1}^{t(N)} \beta_{\eta_i^T} u_{D(\eta_i^T)} - \sum_{i < j} \min\{\beta_{\eta_i^T}, \beta_{\eta_j^T}\} u_{D(\eta_i^T \cup \eta_j^T)} + \dots + \right. \\ & \quad \left. + (-1)^{t(N)-1} \min_{i=1, \dots, t(N)} \{\beta_{\eta_i^T}\} u_{D(\cup_{i=1}^{t(N)} \eta_i^T)} \right)(S) = \\ & = \left(\sum_{i=1}^{t(S)} \beta_{\eta_i^{T,S}} u_{D(\eta_i^{T,S})} - \sum_{i < j} \min\{\beta_{\eta_i^{T,S}}, \beta_{\eta_j^{T,S}}\} u_{D(\eta_i^{T,S} \cup \eta_j^{T,S})} + \dots + \right. \\ & \quad \left. + (-1)^{t(S)-1} \min_{i=1, \dots, t(S)} \{\beta_{\eta_i^{T,S}}\} u_{D(\cup_{i=1}^{t(S)} \eta_i^{T,S})} \right)(S) = \end{aligned}$$

⁴ As we consider the minimal connection graphs of T in (N, γ) , we simplify the notation using η_i^T instead of $\eta_i^{T,N}$

$$\begin{aligned}
&= \sum_{i=1}^{t(S)} \beta_{\eta_i^{T,S}} - \sum_{i < j} \min\{\beta_{\eta_i^{T,S}}, \beta_{\eta_j^{T,S}}\} + \cdots + (-1)^{t(S)-1} \min_{i=1, \dots, t(S)} \{\beta_{\eta_i^{T,S}}\} = \\
&= \sum_{i=1}^{t(S)} \beta_{\eta_i^{T,S}} - \sum_{i=1}^{t(S)-1} \binom{t(S)-i}{1} \beta_{\eta_i^{T,S}} + \sum_{i=1}^{t(S)-2} \binom{t(S)-i}{2} \beta_{\eta_i^{T,S}} + \cdots + (-1)^{t(S)-1} \sum_{i=1}^1 \binom{t(S)-i}{t(S)-1} \beta_{\eta_i^{T,S}} = \\
&= \beta_{\eta_1^{T,S}} \sum_{j=0}^{t(S)-1} (-1)^j \binom{t(S)-1}{j} + \beta_{\eta_2^{T,S}} \sum_{j=0}^{t(S)-2} (-1)^j \binom{t(S)-2}{j} + \cdots + \beta_{\eta_{t(S)-1}^{T,S}} \sum_{j=0}^1 (-1)^j \binom{1}{j} + \beta_{\eta_{t(S)}^{T,S}} = \\
&= \beta_{\eta_{t(S)}^{T,S}} = \max_{i=1, \dots, t(S)} \beta_{\eta_i^{T,S}} = \max_{i=1, \dots, t(S)} \min_{l \in \eta_i^{T,S}} \{w_l\},
\end{aligned}$$

$$\text{as } \sum_{j=0}^{t(S)-k} (-1)^j \binom{t(S)-k}{j} = 0, \text{ for all } k = 1, \dots, t(S) - 1.$$

The proof of *ii*) and *iii*) is trivial. ■

The proof of the next propositions follows the steps of the previous one and then it is omitted.

Proposition 3.2 *Let $(N, u_T, \gamma_w) \in \mathcal{WCS}^{N,d}$, $T \subseteq N$. If $\mathcal{MCG}(T, N, \gamma) = \{\eta_1^T, \dots, \eta_{t(N)}^T\}$ and $\beta_{\eta_i^T} = \frac{1}{1 + \sum_{l \in \eta_i^T} w_l}$ for $i = 1, \dots, t(N)$, then*

$$\begin{aligned}
i) \quad u_T^{\hat{\gamma}^w} &= \sum_{i=1}^{t(N)} \beta_{\eta_i^T} u_{D(\eta_i^T)} - \sum_{i < j} \min\{\beta_{\eta_i^T}, \beta_{\eta_j^T}\} u_{D(\eta_i^T \cup \eta_j^T)} + \cdots + \\
&\quad + (-1)^{t(N)-1} \min_{i=1, \dots, t(N)} \{\beta_{\eta_i^T}\} u_{D(\cup_{i=1}^{t(N)} \eta_i^T)}, \text{ if } |T| > 1 \text{ and } \mathcal{MCG}(T, N, \gamma) \neq \emptyset
\end{aligned}$$

$$ii) \quad u_T^{\hat{\gamma}^w} = 0, \text{ if } |T| > 1 \text{ and } \mathcal{MCG}(T, N, \gamma) = \emptyset$$

$$iii) \quad u_T^{\hat{\gamma}^w} = u_T \text{ if } |T| = 1$$

Proposition 3.3 *Let $(N, u_T, \gamma_w) \in \mathcal{WCS}^{N,f}$, $T \subseteq N$. If $\mathcal{MCG}(T, N, \gamma) = \{\eta_1^T, \dots, \eta_{t(N)}^T\}$ and $\beta_{\eta_i^T} = \frac{1}{1 + \max_{l \in \eta_i^T} w_l}$ for $i = 1, \dots, t(N)$, then*

$$\begin{aligned}
i) \quad u_T^{\hat{\gamma}^w} &= \sum_{i=1}^{t(N)} \beta_{\eta_i^T} u_{D(\eta_i^T)} - \sum_{i < j} \min\{\beta_{\eta_i^T}, \beta_{\eta_j^T}\} u_{D(\eta_i^T \cup \eta_j^T)} + \cdots + \\
&\quad + (-1)^{t(N)-1} \min_{i=1, \dots, t(N)} \{\beta_{\eta_i^T}\} u_{D(\cup_{i=1}^{t(N)} \eta_i^T)}, \text{ if } |T| > 1 \text{ and } \mathcal{MCG}(T, N, \gamma) \neq \emptyset
\end{aligned}$$

$$ii) \quad u_T^{\hat{\gamma}^w} = 0, \text{ if } |T| > 1 \text{ and } \mathcal{MCG}(T, N, \gamma) = \emptyset$$

$$iii) \quad u_T^{\hat{\gamma}^w} = u_T \text{ if } |T| = 1$$

Remark 3.3 *The Myerson graph restricted game can be viewed in the present context as the special case in which all the weighs are equal to one, if we interpret weights as capacities or intimacies, or equal to zero if we interpret*

them as costs or flows. Then, a direct consequence of previous propositions is that, when R is connected in (N, γ_w) and $w_l = 1$ for all $l \in \gamma$ ($w_l = 0$ for all $l \in \gamma$, respectively), every coalition $T \subseteq R$ receives the total dividend $\Delta_v(T)$ (i.e., $\alpha_T^R = 1$) in the weighted-graph restricted game. And thus :

$$\begin{aligned} v^{\gamma_w}(S) &= \sum_{R \in S|\gamma} v^{\gamma_w}(R) = \sum_{R \in S|\gamma} \sum_{T \subseteq R} \Delta_v(T) \alpha_T^R(\{w_l\}) = \sum_{R \in S|\gamma} \sum_{T \subseteq R} \Delta_v(T) = \\ &= \sum_{R \in S|\gamma} v(R) = v^\gamma(S) \end{aligned}$$

for all $S \subseteq N$, and then, in these cases the weighted-graph restricted game coincides with the classical graph restricted game (Myerson, 1977).

Remark 3.4 If (N, γ_w) is a weighted network in which, for each $l \in \gamma$, w_l is interpreted as the probability of a direct relation between its two incident nodes, and these probabilities are considered as independent (as in Calvo et al., 1999) then, for R connected in γ and $T \subseteq R$:

$$\alpha_T^R = \sum_{i=1}^{t(R)} \prod_{l \in \eta_i^{T,R}} w_l - \sum_{i < j} \prod_{l \in \eta_i^{T,R} \cup \eta_j^{T,R}} w_l + \dots + (-1)^{t(R)+1} \prod_{l \in \bigcup_{j=1}^{t(R)} \eta_j^{T,R}} w_l,$$

for $|T| \geq 1$, and $\alpha_T^R = 1$, otherwise.

In this case, α_T^R can be interpreted as the probability of connecting T with (independent) links between members of R selected with the described random scheme.

3.3 Myerson values for weighted-graph restricted games

Definition 3.4 An allocation rule ψ on \mathcal{WCS}^N is a map $\psi : \mathcal{WCS}^N \rightarrow \mathbb{R}^n$, $\psi_i(N, v, \gamma_w)$ representing the outcome for player i in game (N, v) given the restrictions in the communication imposed by the weighted graph (N, γ_w) .

Definition 3.5 The Myerson value for c -weighted communication situations is the allocation rule ψ^c defined on $\mathcal{WCS}^{N,c}$ as $\psi^c(N, v, \gamma_w) = Sh(N, v^{\gamma_w})$

Analogously we can define the Myerson value ψ^d for d -weighted communication situation, the Myerson value ψ^f for f -weighted communication situations and, of course, ψ^p that coincides with the probabilistic Myerson value of Calvo et al. (1999) for p -weighted communication situations.

Remark 3.5 The classical Myerson value is the restriction of ψ^c to the set of those c -weighted communication situations in which the capacities are all equal to 1. Similarly, for ψ^d (ψ^f) and distances or costs (flows) equal to zero.

3.4 Some Examples

Example 3.1 (*Weights are distances or costs*).

Let us consider the following weighted communication situation (N, v, γ_w) in which $N = \{1, 2, 3, 4, 5\}$, $v = u_{\{1,3\}}$ and γ_w is given in the Figure 1.

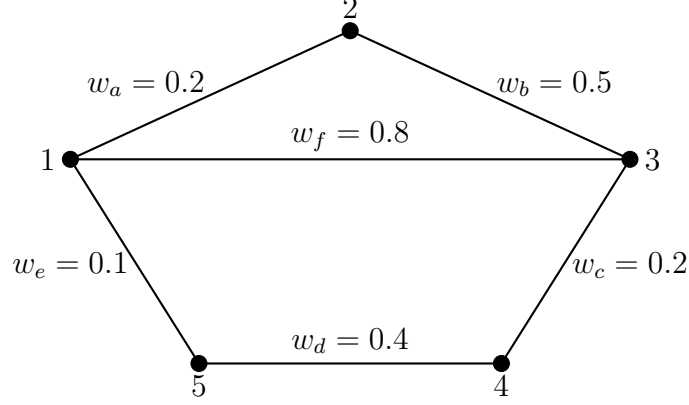


Figure 1.

We will assume that link weights are distances or costs and then, we will calculate $\mu^d(N, v, \gamma_w)$. We have that $\mathcal{MCG}(N, \{1, 3\}, \gamma_w) = \{\eta_1 = \{a, b\}, \eta_2 = \{c, d, e\}, \eta_3 = \{f\}\}$ and thus:

$$\begin{aligned} u_{\{13\}}^{\gamma_w} &= \beta_{\eta_1} u_{D(\eta_1)} + \beta_{\eta_2} u_{D(\eta_2)} + \beta_{\eta_3} u_{D(\eta_3)} - \min\{\beta_{\eta_1}, \beta_{\eta_2}\} u_{D(\eta_1 \cup \eta_2)} - \min\{\beta_{\eta_1}, \beta_{\eta_3}\} u_{D(\eta_1 \cup \eta_3)} - \\ &- \min\{\beta_{\eta_2}, \beta_{\eta_3}\} u_{D(\eta_2 \cup \eta_3)} + \min\{\beta_{\eta_1}, \beta_{\eta_2}, \beta_{\eta_3}\} u_{D(\eta_1 \cup \eta_2 \cup \eta_3)} = \frac{1}{1 + 0.7} u_{\{123\}} + \frac{1}{1 + 0.7} u_{\{1345\}} + \\ &+ \frac{1}{1 + 0.8} u_{\{13\}} - \frac{1}{1 + 0.7} u_{\{12345\}} - \frac{1}{1 + 0.8} u_{\{123\}} - \frac{1}{1 + 0.8} u_{\{1345\}} + \frac{1}{1 + 0.8} u_{\{12345\}} \end{aligned} \quad ^5$$

and then

$$\mu^d(N, v, \gamma_w) = Sh(N, u_{\{13\}}^{\gamma_w}) = (0.2903, 0.0044, 0.2903, 0.0016, 0.0016).$$

Example 3.2 (*Weights are capacities*).

⁵ We simplify the notation writing $\{13\}$ instead of $\{1,3\}$ and so on. Moreover, we can see here a difference with the case of the Myerson restricted game. In that case, if there are several alternative paths to connect 1 and 3 but these nodes are directly connected, all the remaining paths are ignored. In this new approach, this occurs only when the direct path is geodesic among all the possible paths. Otherwise, the nodes in shorter paths are rewarded because their contribution to the communication shortening the distance between 1 and 3

Consider the following weighted communication situation $(N, v, \gamma_w) \in \mathcal{WCS}^{N,c}$ where $N = \{1, 2, 3, 4, 5\}$, $v = u_{\{1,3\}}$ and γ_w is given in the Figure 2.

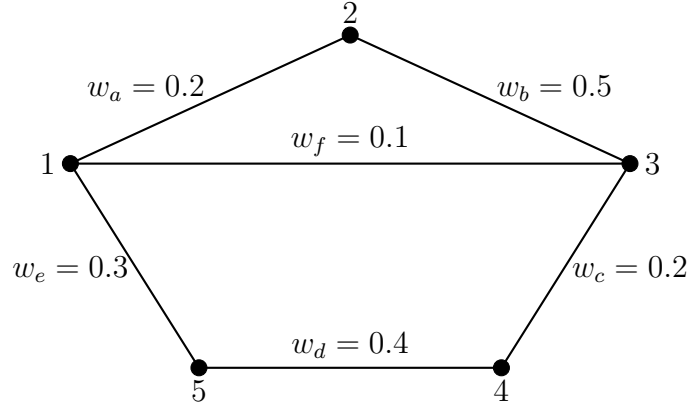


Figure 2.

We will calculate $\mu^c(N, v, \gamma_w)$, i.e., the value of different players when the game is to connect players 1 and 3 taking into account the capacities of the existing communication channels. Of course, $\mathcal{MCG}(N, \{1, 3\}, \gamma_w)$ coincides with the one in the previous example and thus:

$$\begin{aligned} u_{\{13\}}^{\gamma_w} &= \beta_{\eta_1} u_{D(\eta_1)} + \beta_{\eta_2} u_{D(\eta_2)} + \beta_{\eta_3} u_{D(\eta_3)} - \min\{\beta_{\eta_1}, \beta_{\eta_2}\} u_{D(\eta_1 \cup \eta_2)} - \min\{\beta_{\eta_1}, \beta_{\eta_3}\} u_{D(\eta_1 \cup \eta_3)} - \\ &- \min\{\beta_{\eta_2}, \beta_{\eta_3}\} u_{D(\eta_2 \cup \eta_3)} + \min\{\beta_{\eta_1}, \beta_{\eta_2}, \beta_{\eta_3}\} u_{D(\eta_1 \cup \eta_2 \cup \eta_3)} = 0.2u_{\{123\}} + 0.2u_{\{1345\}} + \\ &+ 0.1u_{\{13\}} - 0.2u_{\{12345\}} - 0.1u_{\{123\}} - 0.1u_{\{1345\}} + 0.1u_{\{12345\}} \end{aligned} \quad ^6$$

and then

$$\mu^c(N, v, \gamma_w) = Sh(N, u_{\{13\}}^{\gamma_w}) = (0.0883, 0.0133, 0.0883, 0.005, 0.005).$$

4 Characterizations of the weighted Myerson values

In this section we introduce two characterizations of the Myersons value for weighted-graphs communication situations that generalize the corresponding ones for the unweighted case.

⁶ We can see here too a difference with the case of the Myerson restricted game. In that case, if there are several alternative paths to connect 1 and 3 but these nodes are directly connected, all the remaining paths are ignored. In this new approach, this occurs only when the direct path has the greatest capacity among all the possible paths (the capacity of a path being the minimum of its links capacities).

Definition 4.1 An allocation rule ψ defined on \mathcal{WCS}^N satisfies component efficiency if, for all $(N, v, \gamma_w) \in \mathcal{WCS}^N$ and for all $C \in N \setminus \gamma$,

$$\sum_{i \in C} \psi_i(N, v, \gamma_w) = v^{\gamma_w}(C) \text{ for all } i \in C.$$

Definition 4.2 An allocation rule ψ defined on \mathcal{WCS}^N satisfies fairness if, for all $(N, v, \gamma_w) \in \mathcal{WCS}^N$ and for all $l = \{i, j\} \in \gamma$,

$$\psi_i(N, v, \gamma_w) - \psi_i(N, v, \gamma_w \setminus \{l\}) = \psi_j(N, v, \gamma_w) - \psi_j(N, v, \gamma_w \setminus \{l\}).$$

Definition 4.3 An allocation rule ψ defined on \mathcal{WCS}^N satisfies the balanced contributions property if, for all $(N, v, \gamma_w) \in \mathcal{WCS}^N$ and for all $i, j \in N$,

$$\psi_i(N, v, \gamma_w) - \psi_i(N, v, \gamma_w^{-j}) = \psi_j(N, v, \gamma_w) - \psi_j(N, v, \gamma_w^{-i}).$$

Theorem 4.1 The Myerson value for c -weighted communication situations, μ^c , is the unique allocation rule on $\mathcal{WCS}^{N,c}$ satisfying component efficiency and fairness.

Proof: First, we will prove that μ^c satisfies component efficiency. Let $(N, v, \gamma_w) \in \mathcal{WCS}^{N,c}$. Given $C \in N/\gamma$ we have

$$\sum_{i \in C} \mu_i^c(N, v, \gamma_w) = \sum_{i \in C} \mu_i^c(C, v|_C, (\gamma_w)|_C) = \sum_{i \in C} Sh_i(C, (v|_C)^{(\gamma_w)|_C}) = (v|_C)^{(\gamma_w)|_C}(C)$$

denoting $(\gamma_w)|_C = (\gamma|_C)_{w|(\gamma|_C)}$ and because of the Shapley value efficiency. But, $(v|_C)^{(\gamma_w)|_C}(C) = (v^{\gamma_w})(C)$.

Second, we will prove that μ^c satisfies fairness. As it is obvious from definition, μ^c is linear in the game, so it is sufficient to prove that μ^c satisfies fairness for weighted communication situation in $\mathcal{WCS}^{N,c}$ of the form: (N, u_T, γ_w) , with $T \subseteq N$. If $\mathcal{MCG}(T, N, \gamma) = \{\eta_1^T, \dots, \eta_{t(N)}^T\}$, then the characteristic function $u_T^{\gamma_w}$ is given by:

$$\begin{aligned} & \sum_{i=1}^{t(N)} \beta_{\eta_i^T} u_{D(\eta_i^T)} - \sum_{i < j} \min\{\beta_{\eta_i^T}, \beta_{\eta_j^T}\} u_{D(\eta_i^T \cup \eta_j^T)} + \dots + \\ & + (-1)^{t(N)-1} \min_{i=1, \dots, t(N)} \{\beta_{\eta_i^T}\} u_{D(\cup_{i=1}^{t(N)} \eta_i^T)} \end{aligned}$$

and thus

$$\begin{aligned} \mu^c(N, u_T, \gamma_w) &= Sh\left(\sum_{i=1}^{t(N)} \beta_{\eta_i^T} u_{D(\eta_i^T)} - \sum_{i < j} \min\{\beta_{\eta_i^T}, \beta_{\eta_j^T}\} u_{D(\eta_i^T \cup \eta_j^T)} + \dots + \right. \\ & \left. + (-1)^{t(N)-1} \min_{i=1, \dots, t(N)} \{\beta_{\eta_i^T}\} u_{D(\cup_{i=1}^{t(N)} \eta_i^T)}\right). \end{aligned}$$

Suppose weighted link $l = \{i_0, j_0\}$ is severed. Then

$$\begin{aligned} \mu^c(N, u_T, \gamma_w \setminus \{l\}) &= Sh\left(\sum_{j=1}^k \beta_{\eta_{i_j}^T} u_{D(\eta_{i_j}^T)} - \sum_{j < r} \min\{\beta_{\eta_{i_j}^T}, \beta_{\eta_{i_r}^T}\} u_{D(\eta_{i_j}^T \cup \eta_{i_r}^T)} + \dots \right. \\ &\quad \left. + (-1)^{k+1} \min_{j=1, \dots, k} \{\beta_{\eta_{i_j}^T}\} u_{D(\cup_{j=1}^k \eta_{i_j}^T)}\right), \end{aligned}$$

with $\mathcal{MCG}(T, N, \gamma \setminus \{l\}) = \{\eta_{i_1}^T, \dots, \eta_{i_k}^T\}$ being a subset of $\mathcal{MCG}(T, N, \gamma)$.

Then the difference

$$\mu^c(N, u_T, \gamma_w) - \mu^c(N, u_T, \gamma_w \setminus \{l\})$$

is a linear combination of the Shapley value of games $u_{D(\eta)}$ with $l = \{i_0, j_0\} \in \eta$. In fact, each η is necessarily a union of minimal connection graphs of T in (N, γ) in which, at least, one of them contains l . Of course, $i_0, j_0 \in D(\eta)$ for all these graphs η and because of the Shapley value symmetry the outcome of both players i_0 and j_0 changes by the same amount.

Reciprocally, let us prove the uniqueness. Consider an allocation rule ψ defined on $\mathcal{WCS}^{N,c}$ and satisfying efficiency and fairness. We must prove that $\psi = \mu^c$. The proof uses induction on $|\gamma|$. If $|\gamma| = 0$, then each $i \in N$ forms a component in (N, γ_w) , and then in (N, γ) . As ψ satisfies efficiency in components, for all $i \in N$, $\psi_i(N, v, \gamma_w) = v^{\gamma_w}(\{i\}) = \mu_i^c(N, v, \gamma_w)$ (last equality holding because μ^c also satisfies component efficiency) and thus both allocation rules coincide.

Suppose now, by the induction hypothesis, that $\psi(N, v, \gamma_w) = \mu^c(N, v, \gamma_w)$ for all weighted communication situations in $\mathcal{WCS}^{N,c}$ with $|\gamma| \leq k$ and consider $(N, v, \gamma_w) \in \mathcal{WCS}^{N,c}$ with $|\gamma| = k + 1$. Let $i \in N$ and let $C(i)$ be the class in the quotient set $N/\gamma_w = N/\gamma = \{C_1, C_2, \dots, C_k\}$ to which i belongs. If $C(i) = \{i\}$, then similar as in the case $|\gamma| = 0$ above, by efficiency in components $\psi_i(N, v, \gamma_w) = v^{\gamma_w}(\{i\}) = \mu_i^c(N, v, \gamma_w)$ and thus both rules coincide in i . Alternatively, suppose that $|C(i)| > 1$ and let $j \in C(i)$, $j \neq i$. By the definition of connected component, there exists a sequence of players $i_1 = i, i_2, i_3, \dots, i_r = j$ with $i_l \in C(i)$ for $l = 1, 2, \dots, r$ and such that $\{i_l, i_{l+1}\} \in \gamma$, for each $l = 1, 2, \dots, r - 1$. As ψ satisfies fairness,

$$\psi_{i_1}(N, v, \gamma_w) - \psi_{i_1}(N, v, \gamma_w \setminus \{i_1, i_2\}) = \psi_{i_2}(N, v, \gamma_w) - \psi_{i_2}(N, v, \gamma_w \setminus \{i_1, i_2\}),$$

and thus:

$$\psi_{i_1}(N, v, \gamma_w) - \psi_{i_2}(N, v, \gamma_w) = \psi_{i_1}(N, v, \gamma_w \setminus \{i_1, i_2\}) - \psi_{i_2}(N, v, \gamma_w \setminus \{i_1, i_2\}).$$

As $|\gamma| \leq k$, using the induction hypothesis,

$$\begin{aligned}\psi_{i_1}(N, v, \gamma_w \setminus \{i_1, i_2\}) &= \mu_{i_1}^c(N, v, \gamma_w \setminus \{i_1, i_2\}) \text{ and} \\ \psi_{i_2}(N, v, \gamma_w \setminus \{i_1, i_2\}) &= \mu_{i_2}^c(N, v, \gamma_w \setminus \{i_1, i_2\})\end{aligned}$$

and therefore:

$$\begin{aligned}& \psi_{i_1}(N, v, \gamma_w) - \psi_{i_2}(N, v, \gamma_w) = \\ &= \psi_{i_1}(N, v, \gamma_w \setminus \{i_1, i_2\}) - \psi_{i_2}(N, v, \gamma_w \setminus \{i_1, i_2\}) = \mu_{i_1}^c(N, v, \gamma_w) - \mu_{i_2}^c(N, v, \gamma_w)\end{aligned}$$

the last equality holding because μ^c satisfies the fairness property. As a consequence, $\psi_{i_1}(N, v, \gamma_w) - \mu_{i_1}^c(N, v, \gamma_w) = \psi_{i_2}(N, v, \gamma_w) - \mu_{i_2}^c(N, v, \gamma_w)$.

Iteratively using this previous reasoning, $\psi_i(N, v, \gamma_w) - \mu_i^c(N, v, \gamma_w) = \psi_j(N, v, \gamma_w) - \mu_j^c(N, v, \gamma_w)$ for $j \in C(i)$ and thus, there exists $h_{C(i)} \in \mathbb{R}$ such that $\psi_j(N, v, \gamma_w) - \mu_j^c(N, v, \gamma_w) = h_{C(i)}$ for all $j \in C(i)$. Then,

$$\begin{aligned}|C(i)|h_{C(i)} &= \sum_{j \in C(i)} [\psi_j(N, v, \gamma_w) - \mu_j^c(N, v, \gamma_w)] = \\ &= \sum_{j \in C(i)} \psi_j(N, v, \gamma_w) - \sum_{j \in C(i)} \mu_j^c(N, v, \gamma_w).\end{aligned}$$

By component efficiency of both rules ψ and μ^c , this last expression is equal to zero and thus, $h_{C(i)} = 0 = \psi_j(N, v, \gamma_w) - \mu_j^c(N, v, \gamma_w)$ for all $j \in C(i)$ and in particular for i , which completes the proof. \blacksquare

Similarly we can prove the following results:

Theorem 4.2 *The Myerson value for d -weighted communication situations, μ^d , is the unique allocation rule on $\mathcal{WCS}^{N,d}$ satisfying component efficiency and fairness.*

Theorem 4.3 *The Myerson value for f -weighted communication situations, μ^f , is the unique allocation rule on $\mathcal{WCS}^{N,f}$ satisfying component efficiency and fairness.*

Theorem 4.4 (Calvo et al., 1999, Th.1, pp. 87-89) *The Myerson value for p -weighted communication situations, μ^p , is the unique allocation rule on $\mathcal{WCS}^{N,p}$ satisfying component efficiency and fairness.*

The values μ^c , μ^d , μ^f and μ^p can be also characterized substituting the fairness property by the balanced contributions one.

Theorem 4.5 *The Myerson value for c -weighted communication situations, μ^c , is the unique allocation rule on $\mathcal{WCS}^{N,c}$ satisfying component efficiency and balanced contributions.*

Proof: It is already proved that μ^c satisfies component efficiency.

As μ^c is linear in the game, we only need to prove that μ^c satisfies balanced contributions for weighted communication situation in $\mathcal{WCS}^{N,c}$ of the form: (N, u_T, γ_w) , with $T \subset N$. If $\mathcal{MCG}(T, N, \gamma) = \{\eta_1^T, \dots, \eta_{t(N)}^T\}$, then the characteristic function $u_T^{\gamma_w}$ is given by:

$$\sum_{i=1}^{t(N)} \beta_{\eta_i^T} u_{D(\eta_i^T)} - \sum_{i < j} \min\{\beta_{\eta_i^T}, \beta_{\eta_j^T}\} u_{D(\eta_i^T \cup \eta_j^T)} + \dots + (-1)^{t(N)+1} \min_{i=1, \dots, t(N)} \{\beta_{\eta_i^T}\} u_{D(\bigcup_{i=1}^{t(N)} \eta_i^T)}$$

and thus

$$\begin{aligned} \mu^c(N, u_T, \gamma_w) = & Sh\left(\sum_{i=1}^{t(N)} \beta_{\eta_i^T} u_{D(\eta_i^T)} - \sum_{i < j} \min\{\beta_{\eta_i^T}, \beta_{\eta_j^T}\} u_{D(\eta_i^T \cup \eta_j^T)} + \dots + \right. \\ & \left. + (-1)^{t(N)+1} \min_{i=1, \dots, t(N)} \{\beta_{\eta_i^T}\} u_{D(\bigcup_{i=1}^{t(N)} \eta_i^T)}\right). \end{aligned}$$

Suppose j_0 becomes isolated, then the difference $\mu_{i_0}^c(N, u_T, \gamma_w) - \mu_{i_0}^c(N, u_T, \gamma_w^{-j_0})$ is a linear combination of the Shapley value of games $u_{D(\eta)}$ with $i_0, j_0 \in D(\eta)$. If i_0 becomes isolated, $\mu_{j_0}^c(N, u_T, \gamma_w) - \mu_{j_0}^c(N, u_T, \gamma_w^{-i_0})$ is the same linear combination of the Shapley value of the same games $u_{D(\eta)}$ with $i_0, j_0 \in D(\eta)$. And by the symmetry of Shapley value these quantities coincide.

The proof of the reciprocal mimics the one in Theorem 4.1 and so it is omitted.

■

Similarly we have:

Theorem 4.6 *The Myerson value for d -weighted communication situations, μ^d , is the unique allocation rule on $\mathcal{WCS}^{N,d}$ satisfying component efficiency and balanced contributions.*

Theorem 4.7 *The Myerson value for f -weighted communication situations, μ^f , is the unique allocation rule on $\mathcal{WCS}^{N,f}$ satisfying component efficiency and balanced contributions.*

Theorem 4.8 (Calvo et al., 1999, remark 4.1, p. 89) *The Myerson value for p -weighted communication situations, μ^p , is the unique allocation rule on $\mathcal{WCS}^{N,p}$ satisfying component efficiency and balanced contributions.*

5 Stability

In this section we deal with the problem of determining the extent to which the defined values satisfy (generalized) stability in the sense that if the underlying

game is superadditive then, when varying the weight of a link, other things been equal, the value of both incident nodes change in the appropriate way.

5.1 Stability of μ^c

Given a weighted communication situation $(N, v, \gamma_w) \in \mathcal{WCS}^{N,c}$ we will associate to the weighted graph (N, γ_w) a set of real numbers x_h and a set of deterministic graphs (N, γ_h) , $h = 0, 1, \dots, r$, where $r \leq |\gamma|$ will be the number of different values among the weights of links in (N, γ_w) . So let us define $x_0 = 0$, $(N, \gamma_0) = (N, \gamma)$ and for $h = 1, 2, \dots, r$

$$x_h = \min_{l \in \gamma_{h-1}} \{w_l\} \text{ and } \gamma_h = \{l \in \gamma / w_l > x_h\}.$$

In the following Lemma we prove that for weighted communication situations in $\mathcal{WCS}^{N,c}$ the weighted-graph restricted game can be written as a positive linear combination of deterministic graph restricted games.

Lemma 5.1 *Given $(N, v, \gamma_w) \in \mathcal{WCS}^{N,c}$ with (N, v) a zero-normalized game, it holds that*

$$v^{\gamma_w} = \sum_{h=0}^{r-1} (x_{h+1} - x_h) v^{\gamma_h},$$

where for $h = 0, 1, \dots, r-1$, v^{γ_h} is the Myerson game associated to the deterministic communication situation (N, v, γ_h) .

Proof: Consider $S \subseteq N$ and $R \in S|\gamma$, then

$$v^{\gamma_w}(R) = \sum_{T \subseteq R} \Delta_v(T) \alpha_T^R,$$

with

$$\alpha_T^R = \max_{i=1, \dots, t(R)} \min_{l \in \eta_i^{T,R}} \{w_l\}, \text{ for } |T| \geq 1,$$

where $\{\eta_1^{T,R}, \eta_2^{T,R}, \dots, \eta_{t(R)}^{T,R}\} = \mathcal{MCG}(T, R, \gamma)$ is the family of minimal connection graphs of $T \subseteq R$ in $(R, \gamma|_R)$. Let $\{\eta_{i_1}^{T,R}, \eta_{i_2}^{T,R}, \dots, \eta_{i_a}^{T,R}\}$ be the subfamily of $\mathcal{MCG}(T, R, \gamma)$ such that

$$\alpha_T^R = \min_{l \in \eta_{i_j}^{T,R}} \{w_l\} \text{ for } j = 1, 2, \dots, a.$$

To determine the coefficient that in $\sum_{h=0}^{r-1} (x_{h+1} - x_h) v^{\gamma_h}(R)$ multiplies the dividend $\Delta_v(T)$, let us consider

$$k = \max_h \left\{ h / \text{it exists } \eta_{i_j}^{T,R} \text{ with } \eta_{i_j}^{T,R} \subseteq \gamma_h \text{ and } \eta_{i_j}^{T,R} \not\subseteq \gamma_{h+1} \text{ for all } j = 1, \dots, a \right\}.$$

Then, that coefficient equals to $w_{k+1} = \min_{l \in \gamma_k} \{w_l\}$ and so we only need to prove that $\alpha_T^R = w_{k+1}$. As it exists $j^* \in \{1, 2, \dots, a\}$ such that $\eta_{i_{j^*}}^{T,R} \subseteq \gamma_k$, we have

$$w_{k+1} = \min_{l \in \gamma_k} \{w_l\} \leq \min_{l \in \eta_{i_{j^*}}^{T,R}} \{w_l\} = \alpha_T^R.$$

But if $w_{k+1} < \alpha_T^R$ then, for all $l \in \eta_{i_{j^*}}^{T,R}$, $w_l > w_{k+1}$ which implies $l \in \eta_{i_{j^*}}^{T,R} \subseteq \gamma_{k+1}$. And this contradiction with the definition of k proves the result. ■

Given a communication situation (N, v, γ) , the graph-restricted game (N, v^γ) inherits the superadditivity of game v (Owen, 1986). So, as a direct consequence of the previous lemma, we obtain the following result.

Corollary 5.1 *If $(N, v, \gamma_w) \in \mathcal{WCS}^{N,c}$ and (N, v) is a superadditive game, then (N, v^{γ_w}) is also superadditive.*

Another consequence of the previous lemma is that we can calculate the Myerson value for c -weighted communication situations in terms of Myerson values of appropriate deterministic communication situations. The proof is straightforward from previous Lemma and then it is omitted.

Proposition 5.1 *Given $(N, v, \gamma_w) \in \mathcal{WCS}^{N,c}$ with (N, v_0) the zero-normalization of (N, v) , it holds that*

$$\mu_i^c(N, v, \gamma_w) = \sum_{h=0}^{r-1} (x_{h+1} - x_h) \mu_i(N, v_0, \gamma_h) + v(\{i\}) \text{ for all } i = 1, \dots, n,$$

where for $h = 0, 1, \dots, r-1$, $\mu(N, v, \gamma_h)$ is the Myerson value associated to the deterministic communication situation (N, v, γ) .

As a consequence of the following two lemmas we can establish the stability of the Myerson value for c -weighted communication situations.

Lemma 5.2 *Given $(N, v, \gamma_w), (N, v, \gamma_{w'}) \in \mathcal{WCS}^{N,c}$ with v a superadditive game and such that it exists $l^* = \{i, j\} \in \gamma$ with $w_{l^*} < w'_{l^*}$ and $w_l = w'_l$ for all $l \in \gamma$, $l \neq l^*$, it holds that*

$$\mu_k^c(N, v, \gamma_w) \leq \mu_k^c(N, v, \gamma_{w'}) \text{ for } k=i, j.$$

Proof: Changing w_{l^*} by $w'_{l^*} > w_{l^*}$ some of the pairs (x_{h+1}, γ_h) for $h = 0, \dots, r-1$ are modified. Suppose that $w_{l^*} = x_t$ with $t < r$ (the case in which the maximum weight, x_r is increased is trivial) and let us consider six different possibilities:

i) Several links have weight equal to w_{l^*} but $w'_{l^*} < x_{t+1}$. In this case, in the sequence (x_{h+1}, γ_h) for $h = 0, 1, \dots, r-1$ the pair (x_{t+1}, γ_t) gives rise to the

pair $(w'_{l^*}, \gamma_t \cup \{l^*\})$ and the (x_{t+1}, γ_t) itself, all other couples remaining equal. And so, for $k = i, j$,

$$\begin{aligned} \mu_k^c(N, v, \gamma_{w'}) - \mu_k^c(N, v, \gamma_w) &= (w'_{l^*} - x_t)\mu_k(N, v, \gamma_t \cup \{l^*\}) + (x_{t+1} - w'_{l^*})\mu_k(N, v, \gamma_t) - \\ &\quad - (x_{t+1} - x_t)\mu_k(N, v, \gamma_t) = (w'_{l^*} - x_t)[\mu_k(N, v, \gamma_t \cup \{l^*\}) - \mu_k(N, v, \gamma_t)] \geq 0 \end{aligned}$$

as for $k = i, j$, $\mu_k(N, v, \gamma_t \cup \{l^*\}) \geq \mu_k(N, v, \gamma_t)$ because of the stability of the Myerson value for deterministic communication situations.

ii) Several links have weight equal to w_{l^*} but $w'_{l^*} = x_{t+1}$. Then, (x_{t+1}, γ_t) transforms in $(x_{t+1}, \gamma_t \cup \{l^*\})$, the other pairs being unchanged and thus, for $k = i, j$,

$$\mu_k^c(N, v, \gamma_{w'}) - \mu_k^c(N, v, \gamma_w) = (x_{t+1} - x_t)[\mu_k(N, v, \gamma_t \cup \{l^*\}) - \mu_k(N, v, \gamma_t)] \geq 0$$

again because of the stability of the Myerson value for deterministic communication situations.

iii) Several links have the same weight w_{l^*} but $x_{t+1} < w'_{l^*} < x_{t+2}$. Then, the two pairs (x_{t+1}, γ_t) and (x_{t+2}, γ_{t+1}) transforms in the three pairs $(x'_{t+1} = x_{t+1}, \gamma'_t = \gamma_t \cup \{l^*\})$, $(x'_{t+2} = w'_{l^*}, \gamma'_{t+1} = \gamma_{t+1} \cup \{l^*\})$ and $(x'_{t+3} = x_{t+2}, \gamma'_{t+2} = \gamma_{t+1})$, the rest of the pairs being transformed one to one. Then, for $k = i, j$,

$$\begin{aligned} \mu_k^c(N, v, \gamma_{w'}) - \mu_k^c(N, v, \gamma_w) &= \\ &= (x'_{t+1} - x'_t)\mu_k(N, v, \gamma'_t) + (x'_{t+2} - x'_{t+1})\mu_k(N, v, \gamma'_{t+1}) + \\ &\quad + (x'_{t+3} - x'_{t+2})\mu_k(N, v, \gamma'_{t+2}) - (x_{t+1} - x_t)\mu_k(N, v, \gamma_t) - \\ &\quad - (x_{t+2} - x_{t+1})\mu_k(N, v, \gamma_{t+1}) = (x_{t+1} - x_t)\mu_k(N, v, \gamma_t \cup \{l^*\}) + \\ &\quad + (w'_{l^*} - x_{t+1})\mu_k(N, v, \gamma_{t+1} \cup \{l^*\}) + (x_{t+2} - w'_{l^*})\mu_k(N, v, \gamma_{t+1}) - \\ &\quad - (x_{t+1} - x_t)\mu_k(N, v, \gamma_t) - (x_{t+2} - x_{t+1})\mu_k(N, v, \gamma_{t+1}) \geq 0 \end{aligned}$$

as, because of the stability of Myerson value for deterministic communication situations, for $k = i, j$,

$$\mu_k(N, v, \gamma_t \cup \{l^*\}) \geq \mu_k(N, v, \gamma_t) \text{ and } \mu_k(N, v, \gamma_{t+1} \cup \{l^*\}) \geq \mu_k(N, v, \gamma_{t+1}).$$

iv) Only one link has weight equal to w_{l^*} and $w'_{l^*} < x_{t+1}$. Then (x_t, γ_{t-1}) change to $(x'_t = w'_{l^*}, \gamma'_{t-1} = \gamma_{t-1})$ and thus, for $k = i, j$,

$$\begin{aligned} \mu_k^c(N, v, \gamma_{w'}) - \mu_k^c(N, v, \gamma_w) &= \\ &= (x'_t - x'_{t-1})\mu_k(N, v, \gamma'_{t-1}) + (x'_{t+1} - x'_t)\mu_k(N, v, \gamma'_t) - \\ &\quad - (x_t - x_{t-1})\mu_k(N, v, \gamma_{t-1}) - (x_{t+1} - x_t)\mu_k(N, v, \gamma_t) = \\ &= (w'_{l^*} - x_{t-1})\mu_k(N, v, \gamma_{t-1}) + (x_{t+1} - w'_{l^*})\mu_k(N, v, \gamma_t) - \end{aligned}$$

$$\begin{aligned}
& -(x_t - x_{t-1})\mu_k(N, v, \gamma_{t-1}) - (x_{t+1} - x_t)\mu_k(N, v, \gamma_t) = \\
& = (w'_{l^*} - x_t)[\mu_k(N, v, \gamma_{t-1}) - \mu_k(N, v, \gamma_t)] \geq 0
\end{aligned}$$

because of $\gamma_t = \gamma_{t-1} \setminus \{l^*\}$ and the stability of Myerson value which implies $\mu_k(N, v, \gamma_{t-1}) \geq \mu_k(N, v, \gamma_t)$ for $k = i, j$.

v) Only one link in γ has weight equal to w_{l^*} and $w'_{l^*} = x_{t+1}$. In this case (x_t, γ_{t-1}) and $(x_{t+1}, \gamma_t = \gamma_{t-1} \setminus \{l^*\})$ merge in $(x'_t = w'_{l^*}, \gamma'_{t-1} = \gamma_{t-1})$ and thus, for $k = i, j$,

$$\begin{aligned}
& \mu_k^c(N, v, \gamma_{w'}) - \mu_k^c(N, v, \gamma_w) = \\
& = (x'_t - x'_{t-1})\mu_k(N, v, \gamma'_{t-1}) - (x_t - x_{t-1})\mu_k(N, v, \gamma_{t-1}) - (x_{t+1} - x_t)\mu_k(N, v, \gamma_t) = \\
& = (x_{t+1} - x_{t-1})\mu_k(N, v, \gamma_{t-1}) - (x_t - x_{t-1})\mu_k(N, v, \gamma_{t-1}) - (x_{t+1} - x_t)\mu_k(N, v, \gamma_{t-1} \setminus \{l^*\}) = \\
& = (x_{t+1} - x_t)[\mu_k(N, v, \gamma_{t-1}) - \mu_k(N, v, \gamma_{t-1} \setminus \{l^*\})].
\end{aligned}$$

which is nonnegative as the stability of the Myerson value implies that $\mu_k(N, v, \gamma_{t-1} \setminus \{l^*\}) \leq \mu_k(N, v, \gamma_{t-1})$ for $k = i, j$.

vi) No other link has weight equal to w_{l^*} and $x_{t+1} < w'_{l^*} < x_{t+2}$. Then, the pairs (x_t, γ_{t-1}) and (x_{t+1}, γ_t) transforms respectively in (x_{t+1}, γ_{t-1}) and $(w'_{l^*}, \gamma_t = \gamma_{t+1} \cup \{l^*\})$ and thus, for $k = i, j$,

$$\begin{aligned}
& \mu_k^c(N, v, \gamma_{w'}) - \mu_k^c(N, v, \gamma_w) = \\
& = (x_{t+1} - x_{t-1})\mu_k(N, v, \gamma_{t-1}) + (w'_{l^*} - x_{t+1})\mu_k(N, v, \gamma'_t) + (x_{t+2} - w'_{l^*})\mu_k(N, v, \gamma_{t+1}) - \\
& - (x_t - x_{t-1})\mu_k(N, v, \gamma_{t-1}) - (x_{t+1} - x_t)\mu_k(N, v, \gamma_t) - (x_{t+2} - x_{t+1})\mu_k(N, v, \gamma_{t+1}) = \\
& = (x_{t+1} - x_{t-1})\mu_k(N, v, \gamma_{t-1}) - [(x_t - x_{t-1})\mu_k(N, v, \gamma_{t-1}) + (x_{t+1} - x_t)\mu_k(N, v, \gamma_t = \gamma_{t-1} \setminus \{l^*\})] + \\
& + (w'_{l^*} - x_{t+1})\mu_k(N, v, \gamma'_t = \gamma_{t+1} \cup \{l^*\}) + (x_{t+2} - w'_{l^*})\mu_k(N, v, \gamma_{t+1}) - (x_{t+2} - x_{t+1})\mu_k(N, v, \gamma_{t+1}) = \\
& = (x_{t+1} - x_t)[\mu_k(N, v, \gamma_{t-1}) - \mu_k(N, v, \gamma_t = \gamma_{t-1} \setminus \{l^*\})] + \\
& + (w'_{l^*} - x_{t+1})[\mu_k(N, v, \gamma_{t+1} \cup \{l^*\}) - \mu_k(N, v, \gamma_{t+1})].
\end{aligned}$$

which is nonnegative because of the stability of Myerson value which implies that for $k = i, j$, $\mu_k(N, v, \gamma_t) = \mu_k(N, v, \gamma_{t-1} \setminus \{l^*\}) \leq \mu_k(N, v, \gamma_{t-1})$ and $\mu_k(N, v, \gamma_{t+1}) \leq \mu_k(N, v, \gamma'_t) = \mu_k(N, v, \gamma_{t+1} \cup \{l^*\})$. ■

Lemma 5.3 *Given $(N, v, \gamma_w), (N, v, (\gamma \cup \{l^*\})_{w'}) \in \mathcal{WCS}^{N,c}$ with $w'_l = w_l$ for all $l \in \gamma$ and v being a superadditive game, then, if $l^* = \{i, j\}$, it holds that*

$$\mu_k^c(N, v, \gamma_w) \leq \mu_k^c(N, v, (\gamma \cup \{l^*\})_{w'}) \text{ for } k=i,j.$$

Proof: The proof is similar to the previous one (but shorter) and so it is omitted. ■

Theorem 5.1 *The Myerson value for c -weighted communication situations, μ^c , is stable, i.e., if the underlying game is superadditive, adding a new weighted link to a weighted graph or increasing the capacity of an existing link the value of both incident nodes does not decrease.*

Proof: The result follows iteratively applying both previous lemmas. ■

5.2 Stability of μ^f

The proof of the stability of μ^f mimics the previous one on the stability of μ^c , but associating now to each graph (N, γ_w) in a communication situation $(N, v, \gamma_w) \in \mathcal{WCS}^{N,f}$ the set of real numbers x_h and the set of deterministic graphs (N, γ_h) for $h = 0, \dots, r$ defined by $x_0 = 0$, $(N, \gamma_0) = (N, \gamma)$ and for $h = 1, \dots, r$

$$x_h = \min_{l \in \gamma_{h-1}} \left\{ \frac{1}{1 + w_l} \right\}$$

and $\gamma_h = \{l \in \gamma / \frac{1}{1+w_l} > x_h\}$. Then, we have:

Lemma 5.4 *Given $(N, v, \gamma_w) \in \mathcal{WCS}^{N,f}$ with (N, v) a zero-normalized game, it holds that*

$$v^{\gamma_w} = \sum_{h=0}^{r-1} (x_{h+1} - x_h) v^{\gamma_h}$$

where for $h = 0, 1, \dots, r-1$, v^{γ_h} is the Myerson game associated to the deterministic communication situation (N, v, γ_h) .

A direct consequence of the previous lemma is the following corollary.

Corollary 5.2 *If $(N, v, \gamma_w) \in \mathcal{WCS}^{N,f}$ and (N, v) is a superadditive game, then (N, v^{γ_w}) is also superadditive.*

Similarly, next proposition is straightforward from previous lemma.

Proposition 5.2 *Given $(N, v, \gamma_w) \in \mathcal{WCS}^{N,f}$ with (N, v_0) the zero-normalization of (N, v) , it holds that*

$$\mu_i^f(N, v, \gamma_w) = \sum_{h=0}^{r-1} (x_{h+1} - x_h) \mu_i(N, v_0, \gamma_h) + v(\{i\}) \text{ for all } i = 1, \dots, n$$

where for $h = 0, 1, \dots, r-1$, $\mu(N, v, \gamma_h)$ is the Myerson value associated to the deterministic communication situation (N, v, γ_h) .

Lemma 5.5 *Given $(N, v, \gamma_w), (N, v, \gamma_{w'}) \in \mathcal{WCS}^{N,f}$ with v a superadditive game and such that it exists $l^* = \{i, j\} \in \gamma$ with $w_{l^*} < w'_{l^*}$ and $w_l = w'_l$ for all $l \in \gamma$, $l \neq l^*$ it holds that*

$$\mu_k^f(N, v, \gamma_w) \geq \mu_k^f(N, v, \gamma_{w'}) \text{ for } k=i, j.$$

Theorem 5.2 *The Myerson value for f -weighted communication situations, μ^f , is stable, i.e., if the underlying game is superadditive, increasing the flow of an existing link, the value of both incident nodes does not increase.*

5.3 Stability of μ^d

Unfortunately, the proof of stability obtained for the cases μ^c and μ^f does not hold for μ^d and it remains as an open problem. Nevertheless, the result is easily proved if we strength the requirement on the game to be almost positive (all dividends non negative).

Theorem 5.3 *The Myerson value for d -weighted communication situations, μ^d , is weakly stable, i.e., if the underlying game is almost positive, increasing the distance or cost of an existing link the value of both incident nodes does not increase.*

Proof: Given $(N, v, \gamma_w) \in \mathcal{WCS}^{N,d}$ the weighted-graph restricted game is defined for each $S \subseteq R$ as::

$$v^{\gamma_w}(S) = \sum_{R \in S|\gamma} v^{\gamma_w}(R)$$

where for all $R \in S|\gamma$,

$$v^{\gamma_w}(R) = \sum_{T \subset R} \Delta_v(T) \alpha_T^R(\{w_l\}),$$

with:

$$\alpha_T^R = \max_{i=1, \dots, t(R)} \left\{ \frac{1}{1 + \sum_{l \in \eta_i^{T,R}} w_l} \right\}, \text{ for } |T| \geq 1, \text{ and } \alpha_T^R = 1, \text{ otherwise.}$$

If the weighted communication situation (N, v, γ_w) becomes $(N, v, \gamma_{w'})$ with $l^* = \{i, j\} \in \gamma$ such that $w'_{l^*} > w_{l^*}$ and $w'_l = w_l$ for all $l \neq l^*$ then, for $k = i, j$, and all $S \subseteq N \setminus \{k\}$, $v^{\gamma_{w'}}(S \cup \{k\}) \leq v^{\gamma_w}(S \cup \{k\})$ and $v^{\gamma_{w'}}(S) = v^{\gamma_w}(S)$ hold and thus:

$$\mu_k^d(N, v, \gamma_{w'}) = Sh_k(N, v^{\gamma_{w'}}) \leq Sh_k(N, v^{\gamma_w}) = \mu_k^d(N, v, \gamma_w).$$

■

6 Conclusions and final remarks

So far we have generalized the Myerson value to the framework in which direct relations among actors are not dichotomous but fuzzy or weighted. As the associated weight of a link can admit different meanings, we have adapted the definition of the graph restricted game to take into account these different interpretations. Then, the Shapley value of these weighted-graph games

is used to obtain point solutions for different players. Classical properties and characterizations of Myerson value are translated to the extent possible. We have analyze the extent to which the stability of the Myerson value for deterministic communication situations holds in this new framework. The stability property is satisfied for weighted communication situations in which the weights represent links capacities, intensity in the relations or flow. But when we assume that weights measure distances or costs, the hypothesis of superadditivity must be strengthened to almost positivity. In the way to obtain these results we have proved that μ^c and μ^f can be calculated in terms of a linear combination of the Myerson values of certain deterministic communication situations.

The obtained results permit us to rank nodes in a weighted graph using a game theoretical approach. For weighted graphs representing social networks with different levels of intimacy in the relations among actors, the defined value can be used to obtain a family of centrality measures for such actors.

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